7.3

The Jacobi and Gauss-Siedel Iterative Techniques

Jacobi's Method

Example 1 The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$
\begin{aligned}
E_1 &: \quad 10x_1 - x_2 + 2x_3 = 6,\\
E_2 &: \quad \cancel{-x_1} + 11x_2 \cancel{- x_3} + 3x_4 = 25, \quad \frac{25 + x_1 + x_3 - 3x_4}{11}\\
E_3 &: \quad \cancel{2x_1} \cancel{- x_2} + 10x_3 \cancel{- x_4} = -11, \quad \frac{-11 - 2x_1 + x_2 + x_4}{10}\\
E_4 &: \quad \cancel{3x_2} \cancel{- x_3} + 8x_4 = 15 \quad \frac{15 - 3x_2 + x_3}{8}\\
&\qquad -3x_2 + x_3
\end{aligned}
$$

Solution We first solve equation E_i for x_i, for each $i = 1, 2, 3, 4$, to obtain

$$
\begin{aligned}
x_1^{k} &= \frac{1}{10}x_2^{k-1} - \frac{1}{5}x_3^{k-1} + \frac{3}{5},\\
x_2^{k} &= \frac{1}{11}x_1^{k-1} + \frac{1}{11}x_3^{k-1} - \frac{3}{11}x_4^{k-1} + \frac{25}{11},\\
x_3^{k} &= -\frac{1}{5}x_1^{k-1} + \frac{1}{10}x_2^{k-1} + \frac{1}{10}x_4^{k-1} - \frac{11}{10},\\
x_4^{k} &= -\frac{3}{8}x_2^{k-1} + \frac{1}{8}x_3^{k-1} + \frac{15}{8}.
\end{aligned}
$$

$x_1^0 \quad x_2^0 \quad x_3^0 \quad x_4^0$

From the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ we have $\mathbf{x}^{(1)}$ given by

$$
\begin{aligned}
x_1^{(1)} &= \phantom{-\frac{1}{5}x_1^{(0)}} \quad \cancel{\frac{1}{10}x_2^{(0)}} - \cancel{\frac{1}{5}x_3^{(0)}} \quad + \frac{3}{5} = \underline{0.6000}, \\
x_2^{(1)} &= \cancel{\frac{1}{11}x_1^{(0)}} \quad + \cancel{\frac{1}{11}x_3^{(0)}} - \cancel{\frac{3}{11}x_4^{(0)}} + \boxed{\frac{25}{11}} = 2.2727, \\
x_3^{(1)} &= -\cancel{\frac{1}{5}x_1^{(0)}} + \cancel{\frac{1}{10}x_2^{(0)}} \quad + \cancel{\frac{1}{10}x_4^{(0)}} - \frac{11}{10} = -1.1000, \\
x_4^{(1)} &= \quad - \cancel{\frac{3}{8}x_2^{(0)}} + \cancel{\frac{1}{8}x_3^{(0)}} \quad + \frac{15}{8} = \underline{1.8750}.
\end{aligned}
$$

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are presented in Table 7.1.

$X^1 = (0.6, 2.27, -1.1, 1.875)$

The Gauss-Seidel Method

Example 3 Use the Gauss-Seidel iterative technique to find approximate solutions to

$$
\begin{aligned}
10x_1 - x_2 + 2x_3 \phantom{{}-x_4} &= 6,\\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25,\\
2x_1 - x_2 + 10x_3 - x_4 &= -11,\\
3x_2 - x_3 + 8x_4 &= 15,
\end{aligned}
$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

Solution The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method, we write the system, for each $k = 1, 2, \ldots$ as

$$
\begin{aligned}
x_1^{(k)} &= \phantom{-\frac{1}{11}x_1^{(k)}} \quad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \phantom{+\frac{1}{10}x_4^{(k-1)}} + \frac{3}{5},\\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \phantom{+\frac{1}{10}x_2^{(k)}} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11},\\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \phantom{+\frac{1}{11}x_3^{(k-1)}} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10},\\
x_4^{(k)} &= \phantom{-\frac{1}{5}x_1^{(k)}} - \frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \phantom{+\frac{1}{10}x_4^{(k-1)}} + \frac{15}{8}.
\end{aligned}
$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

☆ write answers as decimal as the final solution
however, when substituting the variable, keep it as fraction

EXERCISE SET 7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b.
$$\begin{aligned} 10x_1 - x_2 \phantom{{}-2x_3} &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ -2x_2 + 10x_3 &= 6. \end{aligned}$$

c.
$$\begin{aligned} 10x_1 + 5x_2 &= 6, \\ 5x_1 + 10x_2 - 4x_3 &= 25, \\ -4x_2 + 8x_3 - x_4 &= -11, \\ -x_3 + 5x_4 &= -11. \end{aligned}$$

d.
$$\begin{aligned} 4x_1 + x_2 + x_3 + x_5 &= 6, \\ -x_1 - 3x_2 + x_3 + x_4 &= 6, \\ 2x_1 + x_2 + 5x_3 - x_4 - x_5 &= 6, \\ -x_1 - x_2 - x_3 + 4x_4 &= 6, \\ 2x_2 - x_3 + x_4 + 4x_5 &= 6. \end{aligned}$$

3. Repeat Exercise 1 using the Gauss-Seidel method.

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.

$$3x_1 - x_2 + x_3 = 1, \implies x_1^k = \frac{1 + x_2^{k-1} - x_3^{k-1}}{3} = \frac{1}{3} + \frac{x_2^{k-1}}{3} - \frac{x_3^{k-1}}{3}$$

$$3x_1 + 6x_2 + 2x_3 = 0, \implies x_2^k = \frac{-3x_1^{k-1} - 2x_3^{k-1}}{6} = -\frac{x_1^{k-1}}{2} - \frac{x_3^{k-1}}{3}$$

$$3x_1 + 3x_2 + 7x_3 = 4. \implies x_3^k = \frac{4 - 3x_1^{k-1} - 3x_2^{k-1}}{7} = \frac{4}{7} - \frac{3x_1^{k-1}}{7} - \frac{3x_2^{k-1}}{7}$$

$$X^0 = 0$$

$$X^0 = (\underset{x_1^0}{0}, \underset{x_2^0}{0}, \underset{x_3^0}{0})^t$$

1st iteration:

$$x_1^1 = \frac{1}{3} + \cancel{\frac{x_2^0}{3}} - \cancel{\frac{x_3^0}{3}} = \frac{1}{3} + \frac{0}{3} - \frac{0}{3} = \frac{1}{3} = 0.33$$

$$x_2^1 = -\cancel{\frac{x_1^0}{2}} - \cancel{\frac{x_3^0}{3}} = -\frac{0}{2} - \frac{0}{3} = 0$$

$$x_3^1 = \frac{4}{7} - \cancel{\frac{3x_1^0}{7}} - \cancel{\frac{3x_2^0}{7}} = \frac{4}{7} = 0.57$$

$$X^1 = (\underset{x_1^1}{0.33}, \underset{x_1^2}{0}, \underset{x_1^3}{0.57})^t$$

2nd iteration:

$$x_1^2 = \frac{1}{3} + \frac{x_2^1}{3} - \frac{x_3^1}{3} = \frac{1}{3} + \frac{0}{3} - \frac{4/7}{3} = \frac{1}{7} = 0.14$$

$$x_2^2 = -\frac{x_1^1}{2} - \frac{x_3^1}{3} = -\frac{1/3}{2} - \frac{4/7}{3} = -\frac{5}{14} = -0.36$$

$$x_3^2 = \frac{4}{7} - \frac{3x_1^1}{7} - \frac{3x_2^1}{7} = \frac{4}{7} - \frac{3(\frac{1}{3})}{7} - \frac{3(0)}{7} = \frac{3}{7} = 0.43$$

$$X^2 = (\underset{x_1^2}{0.14}, \underset{x_2^2}{-0.36}, \underset{x_3^2}{0.43})^t$$

3. Repeat Exercise 1 using the Gauss-Seidel method.

a.

$$3x_1 - x_2 + x_3 = 1,$$
$$3x_1 + 6x_2 + 2x_3 = 0,$$
$$3x_1 + 3x_2 + 7x_3 = 4.$$

$$\Rightarrow X_1^K = \frac{1 + X_2^{K-1} - X_3^{K-1}}{3} = \frac{1}{3} + \frac{X_2^{K-1}}{3} - \frac{X_3^{K-1}}{3}$$

$$\Rightarrow X_2^K = \frac{-3X_1^K - 2X_3^{K-1}}{6} = -\frac{X_1^K}{2} - \frac{X_3^{K-1}}{3}$$

$$\Rightarrow X_3^K = \frac{4 - 3X_1^K - 3X_2^K}{7} = \frac{4}{7} - \frac{3X_1^K}{7} - \frac{3X_2^K}{7}$$

$X^0 = 0$

$X^0 = (0, 0, 0)^t$, with X_1^0, X_2^0, X_3^0

1st iteration:

$$X_1^1 = \frac{1}{3} + \cancel{\frac{X_2^0}{3}} - \cancel{\frac{X_3^0}{3}} = \frac{1}{3} + \frac{0}{3} - \frac{0}{3} = \frac{1}{3} = 0.33$$

$$X_2^1 = -\frac{X_1^1}{2} - \cancel{\frac{X_3^0}{3}} = -\frac{1/3}{2} - \frac{0}{3} = -\frac{1}{6} = -0.167$$

$$X_3^1 = \frac{4}{7} - \frac{3X_1^1}{7} - \frac{3X_2^1}{7} = \frac{4}{7} - \frac{3(\frac{1}{3})}{7} - \frac{3(-\frac{1}{6})}{7} = \frac{1}{2} = 0.5$$

$X^1 = (0.33, -0.167, 0.5)^t$, with X_1^1, X_1^2, X_1^3

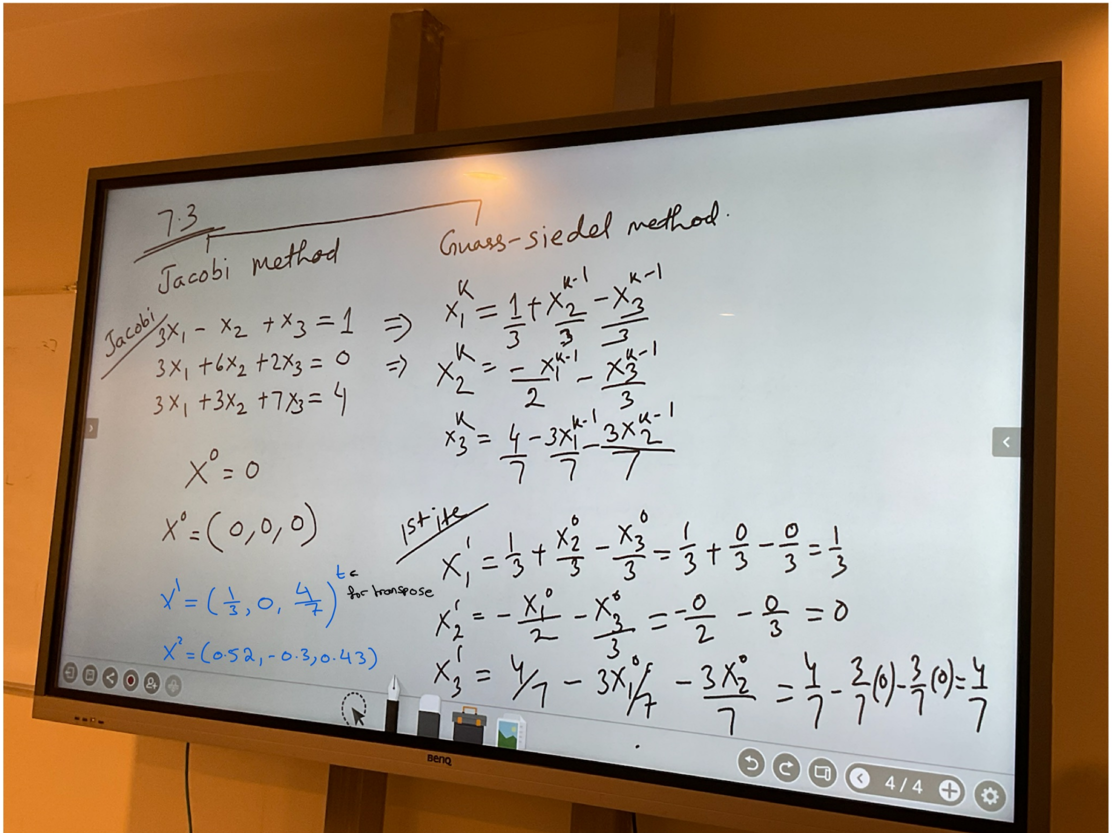
2nd iteration:

$$X_1^2 = \frac{1}{3} + \frac{X_2^1}{3} - \frac{X_3^1}{3} = \frac{1}{3} + \frac{-1/6}{3} - \frac{1/2}{3} = \frac{1}{9} = 0.11$$

$$X_2^2 = -\frac{X_1^2}{2} - \frac{X_3^1}{3} = -\frac{1/9}{2} - \frac{1/2}{3} = -\frac{2}{9} = -0.22$$

$$X_3^2 = \frac{4}{7} - \frac{3X_1^2}{7} - \frac{3X_2^2}{7} = \frac{4}{7} - \frac{3(\frac{1}{9})}{7} - \frac{3(-\frac{2}{9})}{7} = \frac{13}{21} = 0.62$$

$X^2 = (0.11, -0.22, 0.62)^t$, with X_1^2, X_2^2, X_3^2



$$x_1^2 = \frac{1}{3} + \frac{x_2^1}{3} - \frac{x_3^1}{3} = \frac{1}{3} + \frac{0}{3} + \frac{4/7}{3} = \frac{11}{21} = 0.52$$

$$x_2^2 = -\frac{x_2^1}{2} - \frac{x_3^1}{3} = -\frac{1}{3}\left(\frac{1}{3}\right) - \frac{1}{3}\left(\frac{4}{7}\right) = \frac{-19}{63} = -0.3$$

$$x_3^2 = \frac{4}{7} - \frac{3x_1^1}{7} - \frac{3x_2^1}{7} = \frac{4}{7} - \frac{3}{7}\left(\frac{1}{3}\right) - \frac{3}{7}(0) = \frac{3}{7} = 0.43$$

EXERCISE SET 7.3

Jacobi (1a)

$x_1^1 = \frac{1}{3} = 0.33,\ x_2^1 = 0,\ x_3^1 = \frac{4}{7}$

$x^1 = (0.33, 0, 0.57)^t$

2nd ite

$x_1^2 = \frac{1}{3} + \frac{x_2^1}{3} + \frac{x_3^1}{3} = \frac{1}{3} + \frac{0}{3} + \frac{4/7}{3} = \frac{11}{21} = 0.52$

$x_2^2 = -\frac{1}{3}x_1^1 - \frac{1}{3}x_3^1 = -\frac{1}{3}\left(\frac{1}{3}\right) - \frac{1}{3}\left(\frac{4}{7}\right) = -\frac{19}{63} = -0.3$

$x_3^2 = \frac{4}{7} - \frac{3x_1^1}{7} - \frac{3x_2^1}{7}$

$= \frac{4}{7} - \frac{3}{7}\left(\frac{1}{3}\right) - \frac{3}{7}(0)$

$= \frac{3}{7} = 0.43$

$x^2 = (0.52, -0.3, 0.43)$

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a.
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b.
$$\begin{aligned} 10x_1 - x_2 \quad &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ -2x_2 + 10x_3 &= 6. \end{aligned}$$

c.
$$\begin{aligned} 10x_1 + 5x_2 \quad &= 6, \\ 5x_1 + 10x_2 - 4x_3 \quad &= 25, \\ -4x_2 + 8x_3 - x_4 &= -11, \\ -x_3 + 5x_4 &= -11. \end{aligned}$$

d.
$$\begin{aligned} 4x_1 + x_2 + x_3 + x_5 &= 6, \\ -x_1 - 3x_2 + x_3 + x_4 &= 6, \\ 2x_1 + x_2 + 5x_3 - x_4 - x_5 &= 6, \\ -x_1 - x_2 - x_3 + 4x_4 &= 6, \\ 2x_2 - x_3 + x_4 + 4x_5 &= 6. \end{aligned}$$

3. Repeat Exercise 1 using the Gauss-Seidel method.

Jacobi

(1c) $\frac{10x_1}{10} = \frac{6}{10} - \frac{5x_2}{10} \Rightarrow x_1^{k} = \frac{3}{5} - \frac{x_2^{k-1}}{2}$

$10x_2 = 25 - 5x_1 + 4x_3 \Rightarrow x_2^{k} = \frac{25}{10} - \frac{5}{10}x_1 + \frac{4}{10}x_3 = \frac{5}{2} - \frac{x_1^{k-1}}{2} + \frac{2x_3^{k-1}}{5}$

$8x_3 = -11 + 4x_2 + x_4 \Rightarrow x_3^{k} = -\frac{11}{8} + \frac{4}{8}x_2 + \frac{x_4}{8} = -\frac{11}{8} + \frac{x_2^{k-1}}{2} + \frac{x_4^{k-1}}{8}$

$5x_4 = -11 + x_3$ $\quad x_4^{k} = -\frac{11}{5} + \frac{x_3^{k-1}}{5}$

$x^0 = (0, 0, 0, 0)$

1st iter

$x_1^1 = \frac{3}{5} - \frac{x_2^0}{2} = \frac{3}{5} - \frac{0}{2} = \frac{3}{5} = 0.6$

$x_2^1 = \frac{5}{2} - \frac{x_1^0}{2} + \frac{2}{5}x_3^0 = \frac{5}{2} - \frac{0}{2} + \frac{2}{5}(0) = \frac{5}{2} = 2.5$

$x_3^1 = -\frac{11}{8} + \frac{x_2^0}{2} + \frac{x_4^0}{8} = -\frac{11}{8} = -1.375$

$x_4^1 = -\frac{11}{5} + \frac{x_3^0}{5} = -\frac{11}{5} = -2.2$

$x^1 = (0.6, 2.5, -1.375, -2.2)$

2nd iter

$x_1^2 = \frac{3}{5} - \frac{x_2^1}{2} = \frac{3}{5} - \frac{2.5}{2} = -0.65$

$x_2^2 = \frac{5}{2} - \frac{x_1^1}{2} + \frac{2}{5}x_3^1 = \frac{5}{2} - \frac{0.6}{2} + \frac{2}{5}(-1.375)$

$= 1.65$