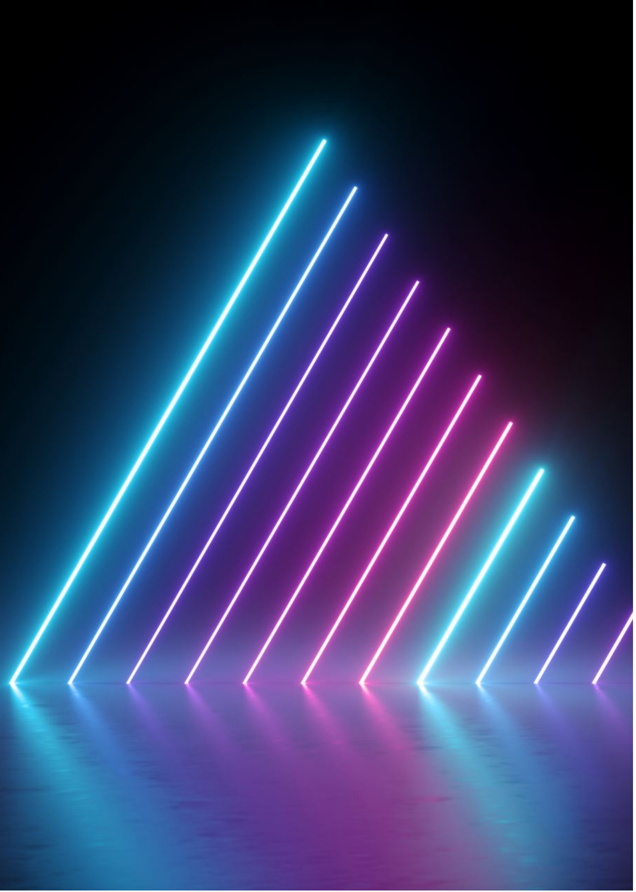

7.2 Eigenvalues and Eigenvectors

If A is a square matrix, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

It is not difficult to show (see Exercise 15) that p is an nth-degree polynomial and, consequently, has at most n distinct zeros, some of which might be complex. If λ is a zero of p, then, since $\det(A - \lambda I) = 0$, Theorem 6.17 on page 402 implies that the linear system defined by $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a solution with $\mathbf{x} \neq \mathbf{0}$. We wish to study the zeros of p and the nonzero solutions corresponding to these systems.

Definition 7.13 If p is the characteristic polynomial of the matrix A, the zeros of p are called **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ. ■

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acteristic values

To determine the eigenvalues of a matrix, we can use the fact that

- λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Once an eigenvalue λ has been found, a corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ is determined by solving the system

- $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

EXERCISE SET 7.2

1. Compute the eigenvalues and associated eigenvectors of the following matrices.

a. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

e. $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$

f. $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

1) a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\underline{A - \lambda I} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$

$(2-\lambda)^2 = 2^2 - 2(2)(\lambda) + \lambda^2$

$\det(A - \lambda I) = \underline{(2-\lambda)(2-\lambda)} - (-1)(-1)$

$= \underline{4} - 4\lambda + \lambda^2 \underline{-1}$

$= \lambda^2 - 4\lambda + 3$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, 3$

For $\lambda = 1$

$R1 + R2$

$\begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} \boxed{1} & -1 \\ 0 & \boxed{0} \end{bmatrix}$

$1 - 1 = 0 \quad -1 + 1 = 0$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$X_1 - X_2 = 0$

$X_1 = X_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1, 1)^t$

For $\lambda = 3$

$-(-1) - 1 = 1 - 1 = 0$

$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{-R1+R2} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-X_1 - X_2 = 0$

$X_1 = -X_2$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1, 1)^t$

$\lambda = 1$ eigen vector is $(1, 1)^t$

$\lambda = 3$ eigen vector is $(-1, 1)^t$

1) f) $A = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

$$\left|\begin{array}{ccc|cc} 2-\lambda & 1 & 1 & 2-\lambda & 1 \\ 2 & 3-\lambda & 2 & 2 & 3-\lambda \\ 1 & 1 & 2-\lambda & 1 & 1 \end{array}\right.$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$\det(A-\lambda I) = (2-\lambda)(3-\lambda)(2-\lambda) + (1)(2)(1) + (1)(2)(1) - (1)(3-\lambda)(1) - (2-\lambda)(2)(1) - (1)(2)(2-\lambda)$$

$$= (2-\lambda)^2(3-\lambda) + 2 + 2 - (3-\lambda) - (4-2\lambda) - (4-2\lambda)$$

$$= (4-4\lambda+\lambda^2)(3-\lambda) + 4 - 3 + \lambda - 4 + 2\lambda - 4 + 2\lambda$$

$$= (12-12\lambda+3\lambda^2-4\lambda+4\lambda^2-\lambda^3) - 7 + 5\lambda$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda = 1, 5$$

For $\lambda=1$

$$A-\lambda I=\begin{bmatrix}2-1 & 1 & 1\\ 2 & 3-1 & 2\\ 1 & 1 & 2-1\end{bmatrix}=\begin{bmatrix}1&1&1\\2&2&2\\1&1&1\end{bmatrix}\xrightarrow[-R_1+R_3]{-2R_1+R_2}\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1+x_2+x_3=0$$

$$x_1=-x_2-x_3$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_2-x_3\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_2\\x_2\\0\end{bmatrix}+\begin{bmatrix}-x_3\\0\\x_3\end{bmatrix}$$

$$=x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

eigenvectors

For $\lambda=5$

$$A-\lambda I=\begin{bmatrix}2-5&1&1\\2&3-5&2\\1&1&2-5\end{bmatrix}=\begin{bmatrix}-3&1&1\\2&-2&2\\1&1&-3\end{bmatrix}\xrightarrow{R1\leftrightarrow R3}\begin{bmatrix}1&1&-3\\2&-2&2\\-3&1&1\end{bmatrix}\xrightarrow[3R1+R3]{-2R1+R2}\begin{bmatrix}1&1&-3\\0&-4&8\\0&4&-8\end{bmatrix}\xrightarrow{-R2+R3}\begin{bmatrix}1&1&-3\\0&-4&8\\0&0&0\end{bmatrix}$$

$-2R_1+R_2$

$3R_1+R_3$

$$\begin{bmatrix}1&1&-3\\0&-4&8\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-4x_2+8x_3=0$$

$$\frac{4x_2}{4}=\frac{8x_3}{4}$$

$$x_2=2x_3$$

$$x_1+x_2-3x_3=0$$

$$x_1+2x_3-3x_3=0$$

$$x_1-x_3=0$$

$$x_1=x_3$$

eigenvector $\begin{bmatrix}1\\2\\1\end{bmatrix}=(1,2,1)^t$

$\lambda=1$ eigenvectors are $(-1,1,0)^t, (-1,0,1)^t$

$\lambda=5$ eigen vector is $(1,2,1)^t$

7.2

(1) (f)

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{matrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{matrix} \right| \begin{matrix} 2-\lambda & 1 \\ 2 & 3-\lambda \\ 1 & 1 \end{matrix}$$

$$= (2-\lambda)(3-\lambda)(2-\lambda) + 2 + 2$$
$$- 2(2-\lambda) - 2(2-\lambda) - (3-\lambda)$$

$$= (2-\lambda)^2(3-\lambda) + \cancel{4} - \cancel{4} + 2\lambda - 4 + 2\lambda - 3 + \lambda$$

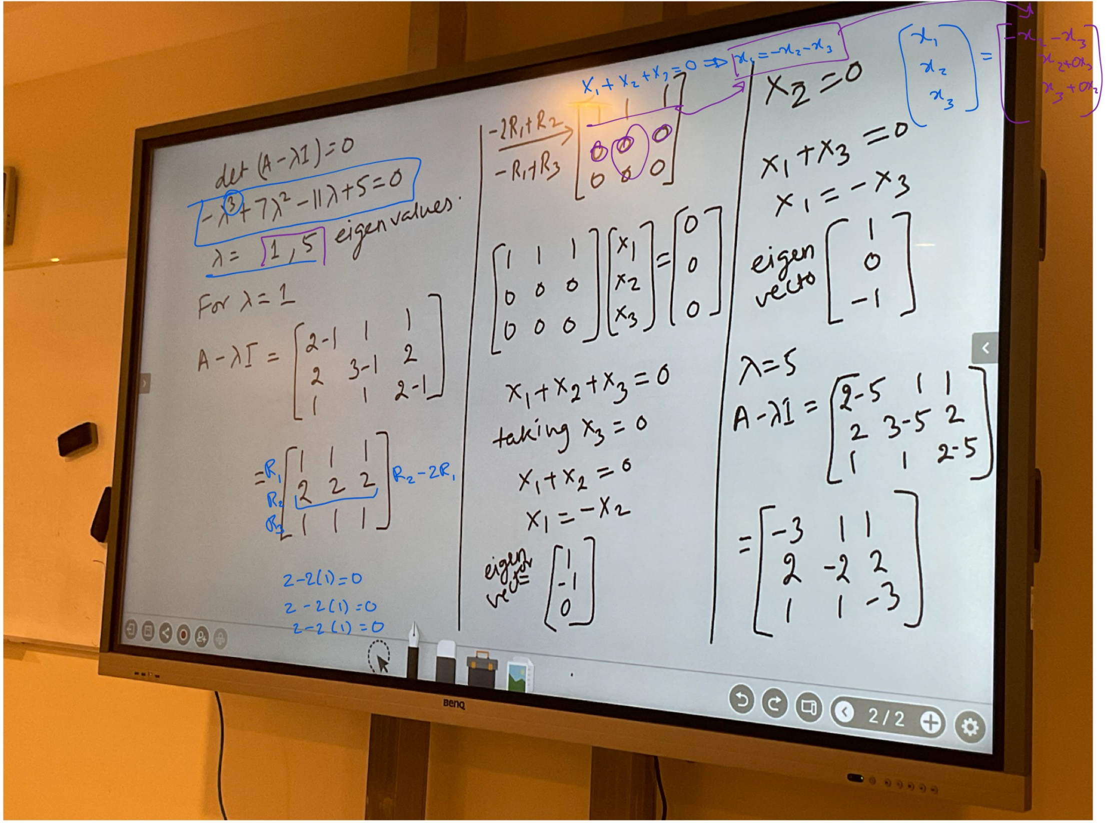
$$= (2-\lambda)^2(3-\lambda) + 5\lambda - 7$$

$$= (4 - 4\lambda + \lambda^2)(3-\lambda) + 5\lambda - 7$$

$$= 12 - 12\lambda + 3\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 + 5\lambda - 7$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

$$\begin{matrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{matrix} \qquad \begin{matrix} x_1 = 2 \\ x_2 = 3 \\ x_3 = 4 \end{matrix}$$



$x_1 + x_2 + x_3 = 0 \Rightarrow \boxed{x_1 = -x_2 - x_3}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 + 0x_3 \\ x_3 + 0x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$

$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

eigen vectors

$X_1 + X_2 + X_3 = 0$

take $X_3 = 0$	take $X_2 = 0$
$X_1 + X_2 = 0$	$X_1 + X_3 = 0$
$X_1 = -X_2$	$X_1 = -X_3$
1st igen Vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$	2nd eigen Vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-4x_2 + 8x_3 = 0 \Rightarrow \frac{4x_2}{4} = \frac{8x_3}{4}$$

$$x_2 = 2x_3$$

$$x_1 + x_2 - 3x_3 = 0$$

$$x_1 + 2x_3 - 3x_3 = 0$$

$$x_1 - x_3 = 0$$

$$x_1 = x_3$$

eigenvector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = (1,2,1)^t$

$\lambda = 1$ eigvectors are

$(1,-1,0)^t, (1,0,-1)^t$ ← transpose

$\lambda = 5$ eigenvector is $(1,2,1)^t$

2 / 2