Definition 6.1

An $\boldsymbol{n} \times \boldsymbol{m}$ ($\boldsymbol{n}$ by $\boldsymbol{m}$) **matrix** is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array. ■

The notation for an $n \times m$ matrix will be a capital letter such as A for the matrix and lowercase letters with double subscripts, such as a_{ij}, to refer to the entry at the intersection of the ith row and jth column; that is,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

Example 1

Determine the size and respective entries of the matrix

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix}$$

Solution The matrix has two rows and three columns so it is of size 2×3. It entries are described by $a_{11} = 2$, $a_{12} = -1$, $a_{13} = 7$, $a_{21} = 3$, $a_{22} = 1$, and $a_{23} = 0$. ■

Solution

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2,\\
\vdots \qquad\qquad & \quad \vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}
$$

by first constructing

$$
A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

$$
[A, \mathbf{b}] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right],
$$

where the vertical dotted line is used to separate the coefficients of the unknowns from the values on the right-hand side of the equations. The array $[A, \mathbf{b}]$ is called an **augmented matrix**.

considering the augmented matrix:

$$\left[\begin{array}{rrrr:r} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right].$$

Performing the operations as described in that example produces the augmented matrices

$$\left[\begin{array}{rrrr:r} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right] \text{ and } \left[\begin{array}{rrrr:r} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right].$$

The final matrix can now be transformed into its corresponding linear system, and solutions for x_1, x_2, x_3, and x_4, can be obtained. The procedure is called **Gaussian elimination with backward substitution**.

Example 2

Represent the linear system

$$
\begin{aligned}
E_1 : \quad x_1 - x_2 + 2x_3 - x_4 &= -8, \\
E_2 : \quad 2x_1 - 2x_2 + 3x_3 - 3x_4 &= -20, \\
E_3 : \quad x_1 + x_2 + x_3 \phantom{{}-3x_4} &= -2, \\
E_4 : \quad x_1 - x_2 + 4x_3 + 3x_4 &= 4,
\end{aligned}
$$

Solution

The augmented matrix is

$$\tilde{A} = \tilde{A}^{(1)} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right]$$

Performing the operations

$$(E_2 - 2E_1) \to (E_2),\ (E_3 - E_1) \to (E_3), \quad \text{and} \quad (E_4 - E_1) \to (E_4),$$

gives

$$\tilde{A}^{(2)} = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right].$$

The diagonal entry $a_{22}^{(2)}$, called the **pivot element**, is 0, so the procedure cannot continue in its present form. But operations $(E_i) \leftrightarrow (E_j)$ are permitted, so a search is made of the elements $a_{32}^{(2)}$ and $a_{42}^{(2)}$ for the first nonzero element. Since $a_{32}^{(2)} \neq 0$, the operation $(E_2) \leftrightarrow (E_3)$ is performed to obtain a new matrix,

$$\tilde{A}^{(2)'} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right].$$

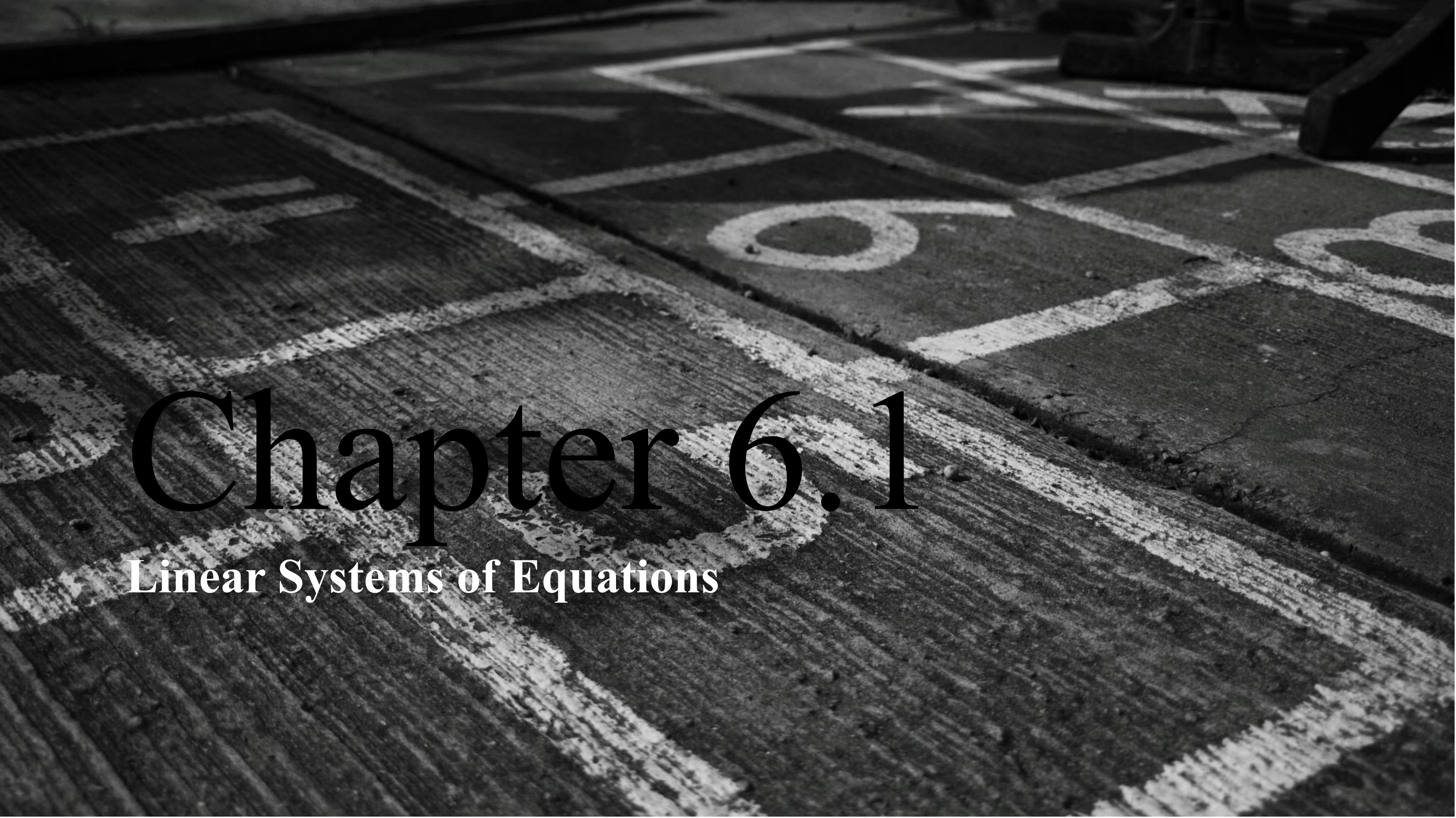
Since x_2 is already eliminated from E_3 and E_4, $\tilde{A}^{(3)}$ will be $\tilde{A}^{(2)'}$, and the computations continue with the operation $(E_4 + 2E_3) \to (E_4)$, giving

$$\tilde{A}^{(4)} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right].$$

Finally, the matrix is converted back into a linear system that has a solution equivalent to the solution of the original system and the backward substitution is applied:

$$
\begin{aligned}
x_4 &= \frac{4}{2} = 2, \\
x_3 &= \frac{[-4 - (-1)x_4]}{-1} = 2, \\
x_2 &= \frac{[6 - x_4 - (-1)x_3]}{2} = 3, \\
x_1 &= \frac{[-8 - (-1)x_4 - 2x_3 - (-1)x_2]}{1} = -7.
\end{aligned}
$$

■



Chapter 6.1

Linear Systems of Equations

EXERCISE SET 6.1

1. For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

a.

$1 + 2(1) = 3$

$$x_1 + 2x_2 = 3,$$
$$x_1 - x_2 = 0.$$

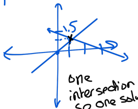
$1 - 1 = 0$

$x_1 + 2x_2 = 3$

x_1	0	1
x_2	1.5	1

$x_1 - x_2 = 0$

x_1	0	1
x_2	0	1



one intersection so one solution

b.

$$x_1 + 2x_2 = 3,$$
$$2x_1 + 4x_2 = 6.$$

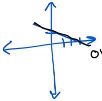
$\times 2$

$x_1 + 2x_2 = 3$

x_1	0	3
x_2	1.5	0

$2x_1 + 4x_2 = 6$

x_1	0	3
x_2	1.5	0



overlapping

c.

$$x_1 + 2x_2 = 0,$$
$$2x_1 + 4x_2 = 0.$$

d.

$$2x_1 + x_2 = -1,$$
$$4x_1 + 2x_2 = -2,$$
$$x_1 - 3x_2 = 5.$$

3. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = 1, x_2 = -1, x_3 = 3$.)

a.

$$4x_1 - x_2 + x_3 = 8,$$
$$2x_1 + 5x_2 + 2x_3 = 3,$$
$$x_1 + 2x_2 + 4x_3 = 11.$$

b.

$$4x_1 + x_2 + 2x_3 = 9,$$
$$2x_1 + 4x_2 - x_3 = -5,$$
$$x_1 + x_2 - 3x_3 = -9.$$

3. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = 1, x_2 = -1, x_3 = 3$.)

a.
$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8, \\ 2x_1 + 5x_2 + 2x_3 &= 3, \\ x_1 + 2x_2 + 4x_3 &= 11. \end{aligned}$$

$-2.3R_2+R_3$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 4 & -1 & 1 & 8 \\ 2 & 5 & 2 & 3 \\ 1 & 2 & 4 & 11 \end{array}\right] \xrightarrow{\frac{R_1}{4}} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 2 & 5 & 2 & 3 \\ 1 & 2 & 4 & 11 \end{array}\right] \xrightarrow{\substack{-2R_1+R_2 \\ -R_1+R_3}} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 5.5 & 1.5 & -1.0 \\ 0 & 2.3 & 3.8 & 9.0 \end{array}\right] \xrightarrow{\frac{1}{5.5}R_2} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 1 & 0.27 & -0.18 \\ 0 & 2.3 & 3.8 & 9.0 \end{array}\right]$$

$$\xrightarrow{-2.3R_2+R_3} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 1 & 0.27 & -0.18 \\ 0 & 0 & 3.2 & 9.4 \end{array}\right]$$

$$\frac{3.2\,x_3}{3.2} = \frac{9.4}{3.2} \qquad \boxed{x_3 = 2.9}$$

$x_2 + 0.27\,x_3 = -0.18$

$x_2 = -0.18 - 0.27x_3$

$x_2 = -0.18 - 0.27(2.9)$

$\boxed{x_2 = -0.96}$

$x_1 - 0.25x_2 + 0.25x_3 = 2$

$x_1 = 2 + 0.25x_2 - 0.25x_3$

$x_1 = 2 + 0.25(-0.96) - 0.25(2.9)$

$\boxed{x_1 = 1.0}$

EXERCISE SET 6.1

5. Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a.
$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2, \\ 3x_1 - 3x_2 + x_3 &= -1, \\ x_1 + x_2 \qquad &= 3. \end{aligned}$$

b.
$$\begin{aligned} 2x_1 - 1.5x_2 + 3x_3 &= 1, \\ -x_1 \qquad + 2x_3 &= 3, \\ 4x_1 - 4.5x_2 + 5x_3 &= 1. \end{aligned}$$

c.
$$\begin{aligned} 2x_1 \qquad\qquad\qquad &= 3, \\ x_1 + 1.5x_2 \qquad\qquad &= 4.5, \\ -\; 3x_2 + 0.5x_3 \qquad &= -6.6, \\ 2x_1 - 2x_2 + x_3 + x_4 &= 0.8. \end{aligned}$$

d.
$$\begin{aligned} x_1 + x_2 \qquad + x_4 &= 2, \\ 2x_1 + x_2 - x_3 + x_4 &= 1, \\ 4x_1 - x_2 - 2x_3 + 2x_4 &= 0, \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3. \end{aligned}$$

5. Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a.
$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2, \\ 3x_1 - 3x_2 + x_3 &= -1, \\ x_1 + x_2 &= 3. \end{aligned}$$

$-3R_1+R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{array}\right] \xrightarrow[-R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & -8 & -7 \\ 0 & 2 & -3 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -8 & -7 \end{array}\right]$$

$-R_1+R_3$

$$\frac{-8x_3}{-8} = \frac{-7}{-8}$$

$$x_3 = \frac{7}{8}$$

$$2x_2 - 3x_3 = 1$$

$$2x_2 - 3\left(\tfrac{7}{8}\right) = 1$$

$$\frac{2x_2}{2} = \frac{1 + \frac{21}{8}}{2}$$

$$x_2 = \frac{29}{16}$$

$$x_1 - x_2 + 3x_3 = 2$$

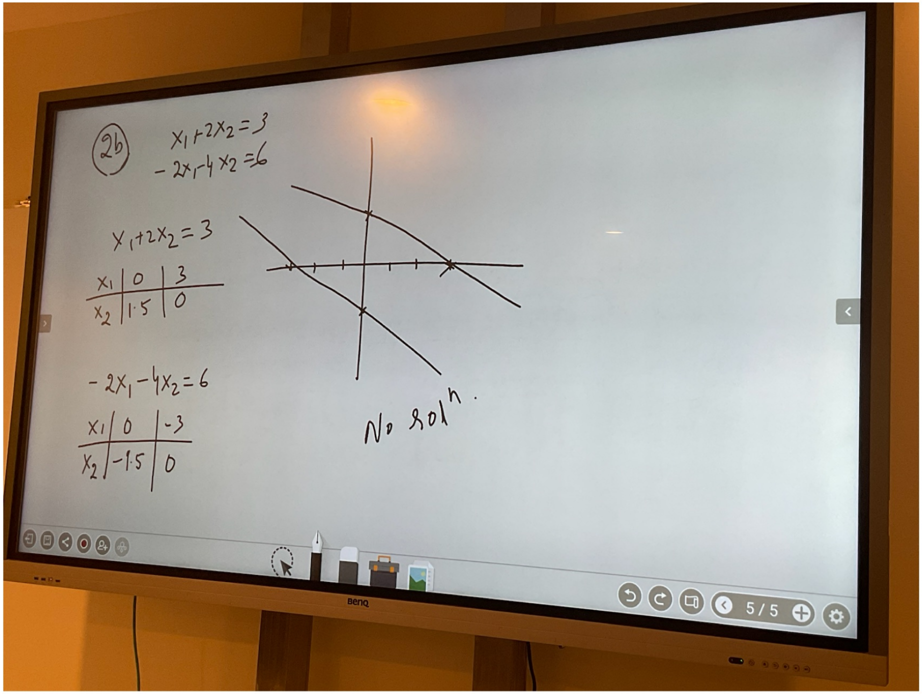
$$x_1 = 2 + x_2 - 3x_3$$

$$x_1 = 2 + \frac{29}{16} - 3\left(\tfrac{7}{8}\right)$$

$$x_1 = \frac{19}{26}$$

$$(x_1, x_2, x_3) = \left(\tfrac{19}{16}, \tfrac{29}{16}, \tfrac{7}{8}\right)$$

with one row exchange



Guassian method

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

3a

$$4x_1 - x_2 + x_3 = 8$$
$$2x_1 + 5x_2 + 2x_3 = 3$$
$$x_1 + 2x_2 + 4x_3 = 11$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & 8 \\ 2 & 5 & 2 & 3 \\ 1 & 2 & 4 & 11 \end{array}\right] \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$$

$$\xrightarrow{\frac{R_1}{4}} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 2.0 & 5.0 & 2.0 & 3.0 \\ 1.0 & 2.0 & 4.0 & 11 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 5.5 & 1.5 & -1.0 \\ 0 & 2.3 & 3.8 & 9.0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5.5}R_2} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 1 & 0.27 & -0.18 \\ 0 & 2.3 & 3.8 & 9.0 \end{array}\right]$$

$$\xrightarrow{-2.3R_2+R_3} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 1 & 0.27 & -0.18 \\ 0 & 0 & 3.2 & 9.4 \end{array}\right]$$

$$-2(+1.0) + 2 = 0$$
$$-2(-0.25) + 5.0 = 5.5$$
$$-2(0.25) + 2 = 1.5$$
$$-2(2) + 3 =$$

$$2 + (-1)(-0.25)$$
$$\underline{2.25}$$

$$-0.25 + 4$$
$$= 3.75$$

$$\left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 5.5 & 1.5 & -1.0 \\ 0 & 2.3 & 3.8 & 9.0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5.5}R_2} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 1.0 & 0.27 & -0.18 \\ 0 & \underline{2.3} & 3.8 & 9.0 \end{array}\right]$$

$$\xrightarrow{-2.3R_2+R_3} \begin{array}{c} \left[\begin{array}{ccc|c} 1.0 & -0.25 & 0.25 & 2.0 \\ 0 & 1.0 & 0.27 & -0.18 \\ 0 & 0 & 3.2 & \cancel{8.6}\ 9.4 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \end{array}$$

$$-2.3(1.0)+2.3=0$$

$$-2.3(0.27)+3.8=3.2$$

$$-2.3(0.27)$$

$$-2.3(-0.18)+9.0=\cancel{8.6}\ 9.4$$

$$(x_1, x_2, x_3) = (1.0, -0.96, 2.9)$$

$$\frac{3.2x_3}{3.2} = \frac{9.4}{3.2}$$

$$x_3 = 2.9$$

$$1.0x_2 + 0.27x_3 = -0.18$$

$$1.0x_2 + 0.27(2.9) = -0.18$$

$$x_2 = -0.18 - 0.27(2.9)$$

$$x_2 = -0.96$$

$$\cancel{1.0x_3 + 0.25x}$$

$$1.0x_1 - 0.25x_2 + 0.25x_3 = 2.0$$

$$x_1 - 0.25(-0.96) + 0.25(2.9) = 2.0$$

$$x_1 = 2.0 + 0.25(-0.96) - 0.25(2.9) = 1.0$$