5.3

Higher-Order Taylor Methods

Formula

Taylor method of order *n*

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1, \tag{5.17}$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i).$$

Euler's method is Taylor's method of order one.

EXERCISE SET 5.3

1. Use Taylor's method of order two to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$
   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with $h = 0.5$
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.25$

3. Repeat Exercise 1 using Taylor's method of order four.

1) a) $f(t,y) = t e^{3t} - 2y$
$y' =$

For order 2: $T^2(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$

$f'(t_i, w_i) = \frac{d}{dt}\left[f(t_i, w_i)\right]$

$uv = u'v + v'u$

$(3t)' = 3$

$\frac{d}{dt}\left[f(t,y)\right] = \frac{d}{dt}\left(te^{3t} - 2y\right)$ $\quad 3e^{3t}$

$= 1e^{3t} + t \cdot 3e^{3t} - 2y'$

$= e^{3t} + 3te^{3t} - 2(te^{3t} - 2y)$

$= e^{3t} + 3te^{3t} - 2te^{3t} + 4y$

$= e^{3t} + te^{3t} + 4y$

$f(t,w) = te^{3t} - 2w$

$f(t_i, w_i) = t_i e^{3t_i} - 2w_i$

$f'(t,w) = e^{3t} + te^{3t} + 4w$

$f'(t_i, w_i) = e^{3t_i} + t_i e^{3t_i} + 4w_i$

$\boxed{T^2(t_i, w_i)} = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$

$= t_i e^{3t_i} - 2w_i + \frac{0.5}{2}\left[e^{3t_i} + t_i e^{3t_i} + 4w_i\right]$

t_i	w_i
$t_0 = 0$	$w_0 = 0$
$t_1 = 0.5$	$w_1 = 0.125$
$t_2 = 1$	$w_2 = 2.623238$

$w_{i+1} = w_i + h\,T^2(t_i, w_i)$

$= w_i + h\left[t_i e^{3t_i} - 2w_i + \frac{h}{2}\left[e^{3t_i} + t_i e^{3t_i} + 4w_i\right]\right]$

$w_1 = w_0 + 0.5\left[t_0 e^{3t_0} - 2w_0 + \frac{0.5}{2}\left[e^{3t_0} + t_0 e^{3t_0} + 4w_0\right]\right]$

$= 0 + 0.5\left[0e^{3(0)} - 2(0) + \frac{0.5}{2}\left[e^{3(0)} + 0e^{3(0)} + 4(0)\right]\right] = 0.125$

$w_2 = w_1 + 0.5\left[t_1 e^{3t_1} - 2w_1 + \frac{0.5}{2}\left[e^{3t_1} + t_1 e^{3t_1} + 4w_1\right]\right]$

$= 0.125 + 0.5\left[0.5e^{3(0.5)} - 2(0.125) + \frac{0.5}{2}\left[e^{3(0.5)} + 0.5e^{3(0.5)} + 4(0.125)\right]\right]$

$= 2.023238$

b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with $h = 0.5$

$f(t,y) = 1 + (t-y)^2$ $\qquad w_0 = 1$

$T^2(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$

$f'(t_i, w_i) = \frac{d}{dt}\left[f(t_i, w_i)\right]$

$\frac{d}{dt}\left[f(t,y)\right] = 2(t-y)(1-y') = 2(t-y)(1-[1+(t-y)^2]) = 2(t-y)(-(t-y)^2) = -2(t-y)(t-y)^2 = -2(t-y)^3$

$f'(t_i, w_i) = -2(t_i - w_i)^3$

$w_{i+1} = w_i + h T^2(t_i, w_i)$

$= w_i + h\left[1 + (t_i - w_i)^2 + \frac{h}{2}\left[-2(t_i - w_i)^3\right]\right]$

$w_1 = w_0 + h\left[1 + (t_0 - w_0)^2 + \frac{h}{2}\left[-2(t_0 - w_0)^3\right]\right]$

$= 1 + 0.5\left[1 + (2-1)^2 + \frac{0.5}{2}\left[-2(2-1)^3\right]\right]$

$= 1.75$

$w_2 = w_1 + h\left[1 + (t_1 - w_1)^2 + \frac{h}{2}\left[-2(t_1 - w_1)^3\right]\right]$

$= 1.75 + 0.5\left[1 + (2.5 - 1.75)^2 + \frac{0.5}{2}\left[-2(2.5-1.75)^3\right]\right]$

$= 2.425781$

t_i	w_i
$t_0 = 2$	$w_0 = 1$
$t_1 = 2.5$	$w_1 = 1.75$
$t_2 = 3$	$w_2 =$

(3d) Taylor's method of order 4

$y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1, \quad h = 0.25$

$w_0 = y(0) = 1.$

$f(t,y) = \cos 2t + \sin 3t \Rightarrow f(t_i, w_i) = \cos 2t_i + \sin 3t_i$

$f'(t,y) = \frac{d}{dt}(\cos 2t + \sin 3t) = -2\sin 2t + 3\cos 3t$

$f''(t,y) = \frac{d}{dt}(-2\sin 2t + 3\cos 3t) = -2(2)\cos 2t + 3(-3)\sin 3t$
$= -4\cos 2t - 9\sin 3t$

$f'''(t,y) = \frac{d}{dt}(-4\cos 2t - 9\sin 3t) = -4(-2)\sin 2t - 9(3)\cos 3t$
$= 8\sin 2t - 27\cos 3t$

$\boxed{n=4}$ $T^4(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + \frac{h^3}{4!} f'''(t_i, w_i)$

$= \cos 2t_i + \sin 3t_i + \frac{h}{2}\left[-2\sin 2t_i + 3\cos 3t_i\right] + \frac{h^2}{6}\left[-4\cos 2t_i - 9\sin 3t_i\right] + \frac{h^3}{24}\left[8\sin 2t_i - 27\cos 3t_i\right]$

$w_{i+1} = w_i + h\, T^4(t_i, w_i)$

$w_{i+1} = w_i + 0.25\left[\cos 2t_i + \sin 3t_i + \frac{0.25}{2}(-2\sin 2t_i + 3\cos 3t_i) + \frac{0.25^2}{6}(-4\cos 2t_i - 9\sin 3t_i) + \frac{(0.25)^3}{24}(8\sin 2t_i - 27\cos 3t_i)\right]$

t_i	w_i
$t_0 = 0$	$w_0 = 1$
$t_1 = 0.25$	$w_1 = 1.32894$
$t_2 = 0.5$	$w_2 = 1.72966$
$t_3 = 0.75$	w_3
$t_4 = 1$	w_4

$w_1 = w_0 + 0.25\left[\cos 2t_0 + \cancelto{0}{\sin 3t_0} + \frac{0.25}{2}(\cancelto{0}{-2\sin 2t_0} + 3\cos 3t_0) + \frac{0.25^2}{6}(-4\cos 2t_0 - \cancelto{0}{9\sin 3t_0}) + \frac{0.25^3}{24}(\cancelto{0}{8\sin 2t_0} - 27\cos 3t_0)\right]$

$w_1 = 1 + 0.25\left[\cos 2(0) + \frac{0.25}{2}(3\cos 3(0)) + \frac{0.25^2}{6}(-4\cos 2(0)) + \frac{0.25^3}{24}(-27\cos 3(0))\right] = 1.32894$

$w_2 = 1.32894 + 0.25\left[\cos 2(0.25) + \sin 3(0.25) + \frac{0.25}{2}(-2\sin 2(0.25) + 3\cos 3(0.25)) + \frac{0.25^2}{6}(-4\cos 2(0.25) - 9\sin 3(0.25)) + \frac{0.25^3}{24}(8\sin 2(0.25) - 27\cos 3(0.25))\right]$

$= 1.72966$

$$w_{i+1} = w_i + h\,T^4(t_i, w_i)$$

$$w_{i+1} = w_i + 0.25\left[\cos 2t_i + \sin 3t_i + \frac{0.25}{2}\left(-2\sin 2t_i + 3\cos 3t_i\right) + \frac{0.25^2}{6}\left(-4\cos 2t_i - 9\sin 3t_i\right) + \frac{(0.25)^3}{24}\left(8\sin 2t_i - 27\cos 3t_i\right)\right]$$

$$w_1 = w_0 + 0.25\left[\cos 2t_0 + \cancelto{0}{\sin 3t_0} + \frac{0.25}{2}\left(-2\cancelto{0}{\sin 2t_0} + 3\cos 3t_0\right) + \frac{0.25^2}{6}\left(-4\cos 2t_0 - 9\cancelto{0}{\sin 3t_0}\right) + \frac{0.25^3}{24}\left(8\cancelto{0}{\sin 2t_0} - 27\cos 3t_0\right)\right]$$

t_i	w_i
$t_0 = 0$	$w_0 = 1$
$t_1 = 0.25$	$w_1 = 1.32894$
$t_2 = 0.5$	$w_2 = 1.72966$
$t_3 = 0.75$	$w_3 = 2.03993$
$t_4 = 1$	$w_4 = 2.11599$

$$w_3 = 1.72966 + 0.25\left[\cos 2(0.5) + \sin 3(0.5) + \frac{0.25}{2}\left(-2\sin 2(0.5) + 3\cos 3(0.5)\right) + \frac{0.25^2}{6}\left(-4\cos 2(0.5) - 9\sin 3(0.5)\right) + \frac{0.25^3}{24}\left(8\sin 2(0.5) - 27\cos 3(0.5)\right)\right] = 2.03993$$

$$w_4 = 2.03993 + 0.25\left[\cos 2(0.75) + \sin 3(0.75) + \frac{0.25}{2}\left(-2\sin 2(0.75) + 3\cos 3(0.75)\right) + \frac{0.25^2}{6}\left(-4\cos 2(0.75) - 9\sin 3(0.75)\right) + \frac{0.25^3}{64}\left(8\cos 2(0.75) - 27\cos 3(0.75)\right)\right]$$

$$= 2.11599$$

3a) $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad h = 0.5, \quad y(0) = 0$

$$f(t,y) = te^{3t} - 2y$$

$$f'(t,y) = 1\cdot e^{3t} + t(3e^{3t}) - 2y' = e^{3t} + 3te^{3t} - 2(te^{3t} - 2y)$$

$$= e^{3t} + 3te^{3t} - 2te^{3t} + 4y = \boxed{e^{3t} + te^{3t} + 4y}$$

$$f''(t,y) = 3e^{3t} + 1\cdot e^{3t} + t(3e^{3t}) + 4y' = 4e^{3t} + 3te^{3t} + 4(te^{3t} - 2y)$$

$$= 4e^{3t} + 3te^{3t} + 4te^{3t} - 8y = \boxed{4e^{3t} + 7te^{3t} - 8y}$$

$$f'''(t,y) = 12e^{3t} + 7(1\cdot e^{3t} + t\,3e^{3t}) - 8y' = 12e^{3t} + 7e^{3t} + 21te^{3t} - 8y'$$

$$= 19e^{3t} + 21te^{3t} - 8(te^{3t} - 2y)$$

$$= 19e^{3t} + 21te^{3t} - 8te^{3t} + 16y = \boxed{19e^{3t} + 13te^{3t} + 16y}$$

$$w_{i+1} = w_i + h\left[f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + \frac{h^3}{4!} f'''(t_i, w_i) \right]$$

$$\begin{aligned} w_2 = w_1 + 0.5 \Big[& t_1 e^{3t_1} - 2w_1 + \frac{0.5}{2}\left(e^{3t_1} + t_1 e^{3t_1} + 4w_1\right) \\ & + \frac{0.5^2}{6}\left(4e^{3t_1} + 7t_1 e^{3t_1} - 8w_1\right) \\ & + \frac{0.5^3}{24}\left(19e^{3t_1} + 13t_1 e^{3t_1} + 16w_1\right) \Big] \end{aligned}$$

$$\begin{aligned} = 0.2578125 + 0.5 \Big[& 0.5e^{3(0.5)} - 2(0.2578125) + \frac{0.5}{2}\left(e^{3(0.5)} + 0.5e^{3(0.5)} + 4(0.2578125)\right) \\ & + \frac{0.5^2}{6}\left(4e^{3(0.5)} + 7(0.5)e^{3(0.5)} - 8(0.2578125)\right) \\ & + \frac{0.5^3}{24}\left(19e^{3(0.5)} + 13(0.5)e^{3(0.5)} + 16(0.2578125)\right) \Big] \end{aligned}$$

$$= 3.055295$$