5.2 Euler's Method

Formula

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1.$$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

Illustration In Example 1, we will use an algorithm for Euler's method to approximate the solution to

$$\underbrace{\boxed{y'}}_{f(t_i,\,w_i)} = y - t^2 + 1, \quad \boxed{0} \le t \le 2, \quad \underbrace{y(0) = 0.5}_{y(0) = w_0},$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

For this problem, $f(t, y) = y - t^2 + 1$; so,

$w_{i+1} = w_i + h\,\underbrace{f(t_i, w_i)}$

$t_0 = 0$

$t_1 = 0 + 0.5 = 0.5$

$t_2 = 0.5 + 0.5 = 1$

$t_3 = 1 + 0.5 = 1.5$

$$w_0 = y(0) = 0.5;$$

$0.5 - 0^2 + 1$

$$w_1 = w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25;$$

$w_0 - t_0^2 + 1$

$$w_2 = w_1 + 0.5\left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25;$$

$$w_3 = w_2 + 0.5\left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375;$$

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375.$$

EXERCISE SET 5.2

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.

   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$

   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with $h = 0.5$

   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$

   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.25$

3. The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the [redacted]

   a. $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

   b. $y(t) = t + \frac{1}{1-t}$

   c. $y(t) = t \ln t + 2t$

   d. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.

a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$

$w_0 = 0$

$w_{i+1} = w_i + hf(t_i, w_i)$

$f(t,y) = te^{3t} - 2y$

$f(t, w) = te^{3t} - 2w$

$f(t_i, w_i) = t_i e^{3t_i} - 2w_i$

$w_{i+1} = w_i + h[t_i e^{3t_i} - 2w_i]$

$w_1 = w_0 + 0.5[0e^{3(0)} - 2w_0] = 0 + 0.5[\cancel{0}(1) - 2(\cancel{0})] = 0 + 0.5[0] = 0$

$w_2 = w_1 + h[t_i e^{3t_i} - 2w_1] = 0 + 0.5[0.5e^{3(0.5)} - 2(0)] = 1.120422$

t_i	w_i
$t_0 = 0$	$w_0 = 0$
$t_1 = 0.5$	$w_1 = 0$
$t_2 = 1$	$w_2 = 1.120422$

c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$

$w_0 = 2$

$f(t, y) = 1 + \frac{y}{t}$

$f(t, w) = 1 + \frac{w}{t}$

$f(t_i, w_i) = 1 + \frac{w_i}{t_i}$

$w_0 = 2$

$w_{i+1} = w_i + hf(t, y) = w_i + h\left[1 + \frac{w_i}{t_i}\right]$

$w_1 = w_0 + 0.25\left[1 + \frac{w_0}{t_0}\right] = 2 + 0.25\left[1 + \frac{2}{1}\right] = 2.75$

$w_2 = w_1 + 0.25\left[1 + \frac{w_1}{t_1}\right] = 2.75 + 0.25\left[1 + \frac{2.75}{1.25}\right] = 3.55$

$w_3 = w_2 + 0.25\left[1 + \frac{w_2}{t_2}\right] = 3.55 + 0.25\left[1 + \frac{3.55}{1.5}\right] = 4.391667$

$w_4 = w_3 + 0.25\left[1 + \frac{w_3}{t_3}\right] = 4.391667 + 0.25\left[1 + \frac{4.391667}{1.75}\right] = 5.269048$

t_i	w_i
$t_0 = 1$	$w_0 = 2$
$t_1 = 1.25$	$w_1 = 2.75$
$t_2 = 1.5$	$w_2 = 3.55$
$t_3 = 1.75$	$w_3 = 4.391667$
$t_4 = 2$	$w_4 = 5.269048$

3) a. $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

t_i	w_i (approximate)	y_i (exact value)	Actual error
$t_0=0$	$w_0=0$	$y_0=0$	0
$t_1=0.5$	$w_1=0$	$y_1=0.283617$	0.283617
$t_2=1$	$w_2=1.120422$	$y_2=3.219099$	2.098677

$$y(0) = \frac{1}{5}(0)e^{3(0)} - \frac{1}{25}e^{3(0)} + \frac{1}{25}e^{-2(0)}$$

$$y(0.5) = \frac{1}{5}(0.5)e^{3(0.5)} - \frac{1}{25}e^{3(0.5)} + \frac{1}{25}e^{-2(0.5)}$$

$$y(1) = \frac{1}{5}(1)e^{3(1)} - \frac{1}{25}e^{3(1)} + \frac{1}{25}e^{-2(1)}$$

3) c. $y(t) = t\ln t + 2t$

$$y(1) = 1\ln 1 + 2(1) = 2$$

$$y(1.25) = 1.25\ln 1.25 + 2(1.25) = 2.7789$$

$$y(1.5) = 1.5\ln 1.5 + 2(1.5) = 3.6081976$$

$$y(1.75) = 1.75\ln 1.75 + 2(1.75) = 4.4723$$

$$y(2) = 2\ln 2 + 2(2) = 5.386294$$

t_i	w_i	y_i	Actual error
$t_0=1$	$w_0=2$	$y_0=2$	0
$t_1=1.25$	$w_1=2.75$	$y_1=2.7789$	0.0289
		$y_2=3.6081976$	0.581976
$t_2=1.5$	$w_2=3.55$	$y_3=4.4793276$	0.0876606
$t_3=1.75$	$w_3=4.391667$		
$t_4=2$	$w_4=5.269048$	$y_4=5.386294$	0.117246