5.1

Initial-Value Problems for Ordinary Differential Equations

Derivative

The Elementary Theory of Initial-Value Problems

If a Lipschitz condition is proved, then the differential equation has a unique solution

Initial condition is present

ex: $y' = 2x + 2$ $y(0) = 2$

Definition 5.1 A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

Lipschitz constant

$$|f(t, y_1) - f(t, y_2,)| \le L|y_1 - y_2|,$$

y is changing

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a **Lipschitz constant** for f. ■

$f(t, y_1) = t|y_1|$ $f(t, y_2) = t|y_2|$ $|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|\,||y_1| - |y_2|| = |t|\,|y_1 - y_2|$ $\le 2\,|y_1 - y_2|$

Example 1 Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) \mid 1 \le t \le 2$ and $-3 \le y \le 4\}$.

L=2 Lipschitz constant

Solution For each pair of points (t, y_1) and (t, y_2) in D, we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|\,||y_1| - |y_2|| \le 2|y_1 - y_2|.$$

Thus, f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2. The smallest value possible for the Lipschitz constant for this problem is $L = 2$ because, for example,

$$|f(2, 1) - f(2, 0)| = |2 - 0| = 2|1 - 0|.$$ ■

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$\frac{\partial}{\partial y}$ derivative with respect to one variable for a multi variable function

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \quad \text{for all } (t, y) \in D, \tag{5.1}$$

delta

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L. ■

EXERCISE SET 5.1

1. Use Theorem 5.4 to show that each of the following initial-value problems has a unique solution and find the solution.

   $f(t,y) = y \cos t$

   a. $y' = y \cos t, \quad 0 \leq t \leq 1, \quad y(0) = 1.$

   b. $y' = \frac{2}{t}y + t^2 e^t, \quad 1 \leq t \leq 2, \quad y(1) = 0.$

   c. $y' = -\frac{2}{t}y + t^2 e^t, \quad 1 \leq t \leq 2, \quad y(1) = \sqrt{2}e.$

   d. $y' = \frac{4t^3 y}{1 + t^4}, \quad 0 \leq t \leq 1, \quad y(0) = 1.$

Condition

1) a) $y' = y\cos t$, $0 \le t \le 1$, $y(0) = 1$

$f(t,y) = y\cos t$

$f(t,y_1) = y_1 \cos t$

$f(t,y_2) = y_2 \cos t$

$|f(t,y_1) - f(t,y_2)| = |y_1 \cos t - y_2 \cos t| = |\cos t|\,|y_1 - y_2|$

$= 1\,|y_1 - y_2|$

$\cos 0 = 1$ $\quad \cos 1 = 0.5$

take the bigger value

$L = 1$, there for it has a unique solution

If L isnt a constant like ∞, it wont have a unique solution

Another way:

$\frac{\partial}{\partial y}(y\cos t)$

$\cos t(1)$

$= \cos(t)$

$\left|\frac{\partial}{\partial y}(y\cos t)\right| = \cos t$

≤ 1

$\cos 0 = 1$ ← bigger

$\cos 1 = 0.5$

Finding solutions:

$dt \times \frac{dy}{dt} = y\cos t \times dt$

$\frac{dy}{y} = \frac{y\cos t\, dt}{y}$ $\qquad \frac{dy}{y} = \cos t\, dt$

$\int \frac{dy}{y} = \int \cos t\, dt$ (seperable method [each variable on one side])

$\int \frac{1}{y}\, dy = \ln y$

$e^{\ln y} = e^{\sin t + C_1}$

$\int \cos t\, dt = \sin t + c$

$y = e^{(\sin t + C_1)}$

$y = e^{\sin t} \cdot e^{C_1}$ $\qquad e^{C_1} = c$

$y(t) = c\, e^{\sin t}$

$y(0) = c\, e^{\sin 0} = 1$ $\qquad e^{\sin 0} = e^0 = 1$

given

$c \Rightarrow 1$ $\qquad c \times 1 = 1$

$c = 1$

$y(t) = e^{\sin t}$

1) d) $y' = \dfrac{4t^3 y}{1+t^4}$ $\quad 0 \leq t \leq 1 \quad y(0)=1$

$$f(t,y) = \frac{4t^3 y}{1+t^4}$$

$$f(t,y_1) = \frac{4t^3}{1+t^4} y_1$$

$$f(t,y_2) = \frac{4t^3}{1+t^4} y_2$$

$$|f(t,y_1) - f(t,y_2)| = \left|\frac{4t^3}{1+t^4} y_1 - \frac{4t^3}{1+t^4} y_2\right| = \left|\frac{4t^3}{1+t^4}\right| |y_1 - y_2|$$

$$\frac{4t^3}{1+t^4} \to t=0 \quad \frac{4(0)^3}{1+0^4} = \frac{0}{1} = 0$$

$$\frac{4t^3}{1+t^4} \to t=1 \quad \frac{4(1)^3}{1+1^4} = \frac{4}{2} = 2$$ ← bigger so $L=2$

therefore there is a unique solution

note: $\ln a + \ln b = \ln(ab)$

$c \ln a = \ln a^c$

$$dt \times \frac{dy}{y\,dt} = \frac{t^3 y}{1+t^4} \cdot \frac{1}{y} \times dt$$

$$\int \frac{dy}{y} = \int \frac{t^3}{1+t^4}\, dt$$

$u = 1+t^4$

$$\frac{d}{dt}(1+t^4) = \frac{d}{dt}(u)$$

$$4t^3 = \frac{du}{dt}$$

$$\frac{4t^3\,dt}{4} = \frac{du}{4}$$

$$\int \frac{t^3}{1+t^4}\, dt$$

$$= \int \frac{du}{4u}$$

$$= \frac{1}{4} \int \frac{du}{u}$$

$$= \frac{1}{4} \ln u$$

$$= \frac{1}{4} \ln(1+t^4)$$

$$\ln y = \frac{1}{4} \ln(1+t^4) + \ln C$$

$$\ln y = \ln(1+t^4)^{\frac{1}{4}} + \ln C$$

$\ln a + \ln c$

$\ln ac$

$$\ln y = \ln(\sqrt[4]{1+t^4})C$$

$$y = \sqrt[4]{1+t^4}\, C$$

$$y(0) = \sqrt[4]{1+0^4} + C = 1 \Rightarrow C = 0$$

$$y = (1+t^4)^{\frac{1}{4}}$$

$$y(t) = (1+t^4)^{\frac{1}{4}} = \sqrt[4]{1+t^4}$$

$$\sqrt[4]{1+0^4} + c = 1$$

$$1 + c = 1$$

$$-1 \quad -1$$

$$c = 0$$