Ch-4.3

ELEMENTS OF NUMERICAL INTEGRATION

The need often arises for evaluating the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain. The basic method involved in approximating $\int_a^b f(x)\,dx$ is called **numerical quadrature**. It uses a sum $\sum_{i=0}^{n} a_i f(x_i)$ to approximate $\int_a^b f(x)\,dx$.

Note that when n is an even integer, the degree of precision is $n + 1$, although the interpolation polynomial is of degree at most n. When n is odd, the degree of precision is only n.

Some of the common **closed Newton-Cotes formulas** with their error terms are listed. Note that in each case the unknown value ξ lies in (a, b).

$h = \frac{x_1 - x_0}{n} = \frac{1-0}{1} = \frac{1}{1} = 1$

***n* = 1: Trapezoidal rule**

$x_0 = 0$

$x_1 = 1$

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)]$$

$= \frac{1}{2}[f(0) + f(1)]$

***n* = 2: Simpson's rule**

$h = \frac{x_2 - x_0}{2} = \frac{1-0}{2} = \frac{1}{2}$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$

$x_1 = x_0 + h = 0 + \frac{1}{2} = \frac{1}{2}$

***n* = 3: Simpson's Three-Eighths rule**

$h = \frac{x_3 - x_0}{3} = \frac{1.5 - 0}{3} = 0.5 = \frac{1}{2}$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

$x_0 = 0$ $x_3 = 1.5$

$x_1 = x_0 + \frac{1}{2} = 0.5$

$x_2 = x_1 + \frac{1}{2} = 0.5 + 0.5 = 1$

***n* = 4:**

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

Open Newton-Cotes Formulas

The *open Newton-Cotes formulas* do not include the endpoints of $[a, b]$ as nodes. They use the nodes $x_i = x_0 + ih$, for each $i = 0, 1, \ldots, n$, where $h = (b - a)/(n + 2)$ and $x_0 = a + h$. This implies that $x_n = b - h$, so we label the endpoints by setting $x_{-1} = a$ and $x_{n+1} = b$, as shown in Figure 4.6. Open formulas contain all the nodes used for the approximation within the open interval (a, b). The formulas become

$$\int_a^b f(x)\,dx = \int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i),$$

where $a_i = \int_a^b L_i(x)\,dx$.

$n = 0$: Midpoint rule

$$h = \frac{x_1 - x_{-1}}{n+2}$$

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0)$$

$n = 1$:

$$x_0 = x_{-1} + h$$

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}[f(x_0) + f(x_1)]$$

if its a repeating number, write a fraction
or nonterminating number

$n = 2$:

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)]$$

$n = 3$:

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{5h}{24}[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)]$$

Exercise Set 4.3

1. Approximate the following integrals using the Trapezoidal rule. $n=1$

a. $\int_{0.5}^{1} x^4\,dx$

b. $\int_{0}^{0.5} \frac{2}{x-4}\,dx$

c. $\int_{1}^{1.5} x^2 \ln x\,dx$ $h = \frac{1.5-1}{1} = \frac{0.5}{1} = 0.5$

d. $\int_{0}^{1} x^2 e^{-x}\,dx$

e. $\int_{1}^{1.6} \frac{2x}{x^2-4}\,dx$

f. $\int_{0}^{0.35} \frac{2}{x^2-4}\,dx$

g. $\int_{0}^{\pi/4} x \sin x\,dx$

h. $\int_{0}^{\pi/4} e^{3x} \sin 2x\,dx$

c. $\int_{1}^{1.5} x^2 \ln x \, dx$ $n=1$

$h = \frac{1.5-1}{1} = 0.5$

$f(x_1) - f(x_0)$

$\frac{0.5}{2}\left[1.5^2 \ln 1.5 + 1^2 \ln 1\right] = \underline{0.2281}$

subtract to find error bound

$\int_{1}^{1.5} x^2 \ln x \, dx = \underline{0.1923}$

5.

Repeat Exercise 1 using Simpson's rule. $n = 2$

c. $\displaystyle\int_{1}^{1.5} x^2 \ln x \, dx = \frac{0.25}{3}\left[1^2 \ln x + 4(1.25^2 \ln 1.25) + (1.5^2 \ln 1.5)\right]$

$= 0.19224$

$h = \frac{1.5 - 1}{2} = 0.25$

$x_0 = 1$

$x_1 = 1 + 0.25 = 1.25$

$x_2 = 1.5$

9.

Open Cotes

$n=0$ midpoint

$n=1$

$n=2$

$n=3$

Closed Cotes

$n=1$ trapezoidal

$n=2$ Simpson

$n=3$ Simpson $\frac{3}{8}$

$n=4$

$h = \frac{1.5-1}{0+2}$

open cotes

Repeat Exercise 1 using the Midpoint rule. $n=0$

c. $\int_{1 = x_{-1}}^{1.5 = x_1} x^2 \ln x \, dx$

$h = \frac{1.5-1}{0+2} = 0.25$

$x_0 = 1 + 0.25 = 1.25$

$x_0 = x_{-1} + h$

$= 2(0.25)(1.25^2 \ln 1.25) = 0.1743$

$2h \times f(x_0)$

0.1923

The Trapezoidal Rule

To derive the Trapezoidal rule for approximating $\int_a^b f(x)\,dx$, let $x_0 = a$, $x_1 = b$, $h = b - a$ and use the linear Lagrange polynomial:

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)]$$

Simpson's Rule

Simpson's rule results from integrating over $[a, b]$ the second Lagrange polynomial equally-spaced nodes $x_0 = a$, $x_2 = b$, and $x_1 = a + h$, where $h = (b - a)/2$.

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$n = 0$: Midpoint rule

$$\int_{x_{-1}}^{x_1} f(x)\, dx = 2h f(x_0)$$

h. $\int_0^{\pi/4} e^{3x} \sin 2x \, dx$

Trapezoidal method

$n=1$ $\quad h = \frac{\frac{\pi}{4}-0}{1} = \frac{\pi}{4}$ $\quad$ exact: 1.359

$$= \frac{h}{2}\left[f(x_0)+f(x_1)\right]$$

$$= \frac{\frac{\pi}{4}}{2}\left[\cancel{e^{3(0)}\sin(2(0))}^{0} + e^{3(\frac{\pi}{4})}\sin 2(\tfrac{\pi}{4})\right]$$

$$= \frac{\pi}{8}\left[e^{3(\frac{\pi}{4})}\sin(2\times\tfrac{\pi}{4})\right]$$

3.

Find a bound for the error in Exercise 1 using the error formula, and compare this to the actual error.

$$\int_{x_0=0.5}^{x_1=1} x^{4}\,dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right] = \frac{0.5}{2}\left[0.5^4+1^4\right] = 0.265625$$

$$h = \frac{1-0.5}{1} = 0.5$$

$$\int_{0.5}^{1} x^4\,dx = \left[\frac{x^5}{5}\right]_{0.5}^{1} = \left[\frac{1^5}{5} - \frac{0.5^5}{5}\right] = 0.19375$$