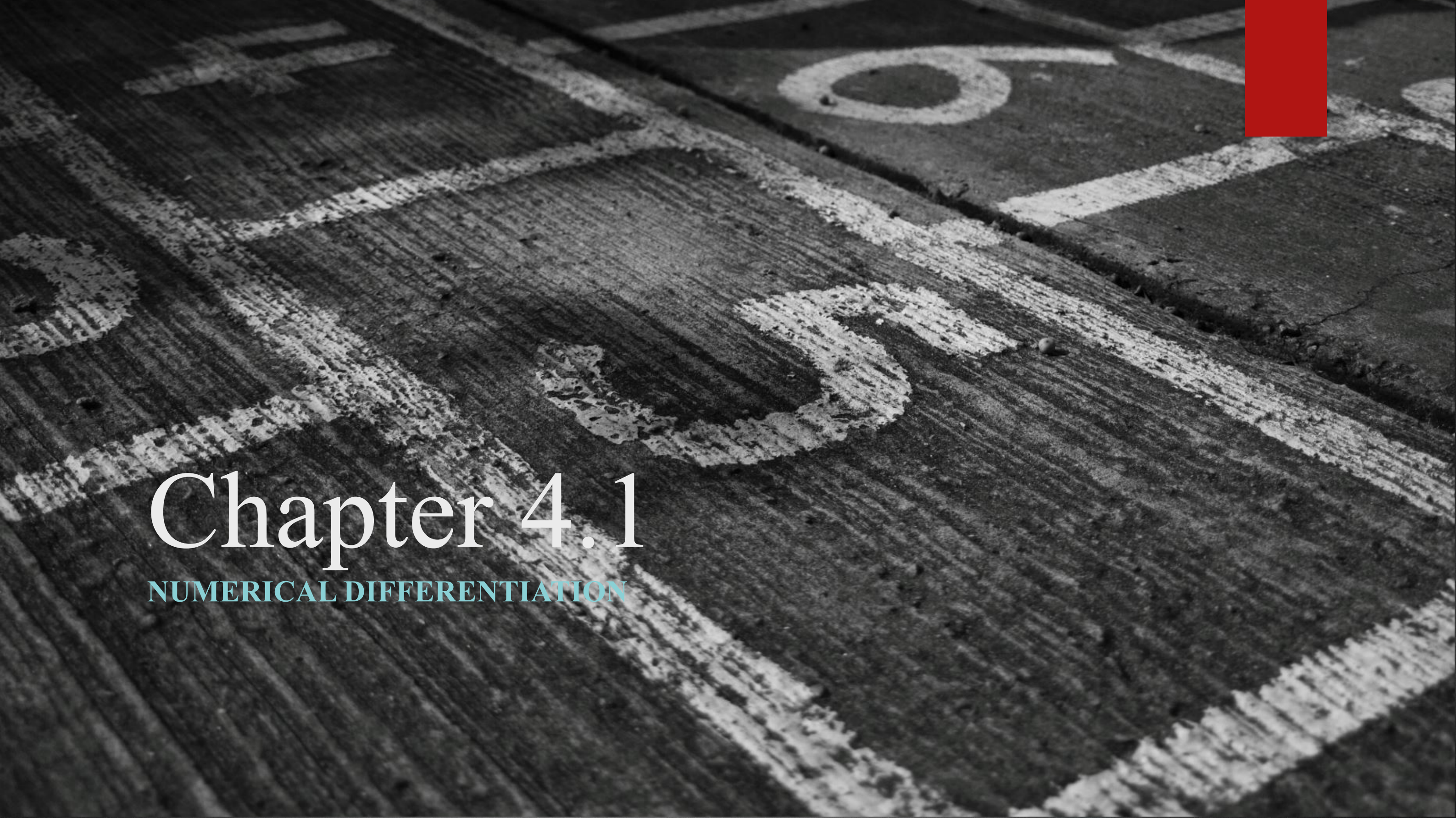

Chapter 4.1

NUMERICAL DIFFERENTIATION

The derivative of the function f at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This formula gives an obvious way to generate an approximation to $f'(x_0)$; simply compute

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

for small values of h. Although this may be obvious, it is not very successful, due to our old nemesis round-off error. But it is certainly a place to start.

Three-Point Endpoint Formula

$-3f(x_0)+4f(x_1)-f(x_2)$

- $$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \cancel{\frac{h^2}{3}f^{(3)}(\xi_0)},$$

~~where ξ_0 lies between x_0 and $x_0 + 2h$.~~

Three-Point Midpoint Formula

$f'(x_1) = \frac{1}{2}[f(x_2) - f(x_0)]$

- $$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(\underline{x_0-h})] \cancel{- \frac{h^2}{6}f^{(3)}(\xi_1)},$$

~~where ξ_1 lies between $x_0 - h$ and $x_0 + h$.~~

$x_0 \quad x_1 \quad x_2$

$\uparrow$

5. Use the most accurate three-point formula to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
1.1	9.025013	17.769705
1.2	11.02318	22.193635
1.3	13.46374	27.10735
1.4	16.44465	32.51085

b.

x	$f(x)$	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

5 a)

3 endpoint (forward): (h is pos)

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

$$f'(1.1) = \frac{1}{2(0.1)}\left[-3f(1.1) + 4f(1.2) - f(1.3)\right]$$

$$= \frac{1}{2(0.1)}\left[-3(9.025013) + 4(11.02318) - 13.46374)\right] = 17.769705$$

$$f'(1.2) = \frac{1}{2(0.1)}\left[-3(11.02318) + 4(13.46374) - 16.44465)\right] \approx 21.70385$$

3 endpoint (backward): (h is neg)

$$f'(1.3) = \frac{1}{2(-0.1)}\left[-3(13.46374) + 4(11.02318) - 9.025013\right] = 26.617565$$

$$f'(1.4) = \frac{1}{2(-0.1)}\left[-3(16.44465) + 4(13.46374) - 11.02318\right] = 32.51085$$

3 midpoint formula: (h is pos)

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$$f'(1.2) = \frac{1}{2(0.1)}\left[f(1.3) - f(1.1)\right]$$

$$= \frac{1}{2(0.1)}\left[13.46374 - 9.025013\right] = 22.193635$$

$$f'(1.3) = \frac{1}{2(0.1)}\left[16.44465 - 11.02318\right] = 27.10735$$

Five-Point Midpoint Formula

$f'(x_3) = \frac{1}{12h}[f(x_1) - 8f(x_2) + 8f(x_3) - f(x_4)]$

- $$f'(x_0) = \frac{1}{12h}[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] + \cancel{\frac{h^4}{30} f^{(5)}(\xi)}, \quad (4.6)$$

~~where ξ lies between $x_0 - 2h$ and $x_0 + 2h$.~~

The derivation of this formula is considered in Section 4.2. The other five-point formula is used for approximations at the endpoints.

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

Five-Point Endpoint Formula

forward+backward

$\underline{x_1}$ x_2 x_3 x_4 x_5

$$f'(x_0) = \frac{1}{12h}[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)] + \cancel{\frac{h^4}{5}f^{(5)}(\xi)},$$

~~where ξ lies between x_0 and $x_0 + 4h$.~~

$$x_1 = \frac{1}{12h}\left[-25f(x_1) + 48f(x_2) - 36f(x_3) + 16f(x_4) - 3f(x_5)\right]$$

Second Derivative Midpoint Formula

- $$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \cancel{\frac{h^2}{12} f^{(4)}(\xi)}, \qquad (4.9)$$

~~for some ξ, where $x_0 - h < \xi < x_0 + h$.~~

If $f^{(4)}$ is continuous on $[x_0 - h, x_0 + h]$ it is also bounded, and the approximation is $O(h^2)$.

9. Use the formulas given in this section to determine, as accurately as possible, approximations for each missing entry in the following tables.

a.

$h=0.1$

x	$f(x)$	$f'(x)$
2.1 x_0	−1.709847	3.8993
2.2 x_1	−1.373823	2.8769
2.3 x_2	−1.119214	2.2497
2.4 x_3	−0.9160143	1.8378
2.5 x_4	−0.7470223	1.5442
2.6 x_5	−0.6015966	1.3555

5 point forward end point method: (needs 4 po. ...d)

$$f'(x_0) = \frac{1}{12h}[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)]$$

$$f'(2.1) = \frac{1}{12(0.1)}\left[-25(-1.709847)+48(-1.37823)-36(-1.119214)+16(-0.9160143)-3(-0.7470223)\right] = 3.8993$$

$$f'(2.2) = \frac{1}{12(0.1)}\left[-25(-1.37823)+48(-1.119214)-36(-0.9160143)+16(-0.7470223)-3(-0.6015966)\right] = 2.8769$$

5 point midpoint: (needs 2 points forward and 2 points backward)

$$f'(x_0) = \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)]$$

$$f'(2.3) = \frac{1}{12(0.1)}\left[-1.709847-8(-1.373823)+8(-0.9160143)-(-0.7470223)\right] = 2.2497$$

$$f'(2.4) = \frac{1}{12(0.1)}\left[-1.373823-8(-1.119214)+8(-0.7470223)-(-0.6015966)\right] = 1.8378$$

5 point backward endpoint method: (needs 4 points backwards and h is negative)

$$f'(x_0) = \frac{1}{12h}[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)]$$

$$f'(2.5) = \frac{1}{12(0.1)}\left[-25(-0.7470223) + 48(-0.9160143) - 36(-1.119214) + 16(-1.37823) - 3(-1.709847)\right] = 1.5442$$

$$f'(2.6) = \frac{1}{12(0.1)}\left[-25(-0.6015966) + 48(-0.7470223) - 36(-0.9160143) + 16(-1.119214) - 3(-1.37823)\right] = 1.3555$$

Exercise Set 4.1

1. Use the forward-difference formulas and backward-difference formulas to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
0.5	0.4794	0.852
0.6	0.5646	0.852
0.7	0.6442	0.796

b.

x	$f(x)$	$f'(x)$
0.0	0.00000	
0.2	0.74140	
0.4	1.3718	

2. Use the forward-difference formulas and backward-difference formulas to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
−0.3	1.9507	
−0.2	2.0421	
−0.1	2.0601	

b.

x	$f(x)$	$f'(x)$
1.0	1.0000	
1.2	1.2625	
1.4	1.6595	

3. The data in Exercise 1 were taken from the following functions. Compute the actual errors in Exercise 1, and find ~~error bounds using the error formulas.~~

only

a. $f(x) = \sin x$

b. $f(x) = e^x - 2x^2 + 3x - 1$

3. The data in Exercise 1 were taken from the following functions. Compute the actual errors in Exercise 1, and find ~~error bounds using the error formulas.~~

only

a. $f(x) = \sin x$ b. $f(x) = e^x - 2x^2 + 3x - 1$

$f'(x) = \cos x$

* Note: put radian on calculator

$f'(0.5) = \cos(0.5) = 0.87758$

Actual error: $|0.877582 - 0.852| = 0.025$

$f'(0.6) = \cos(0.6) = 0.825$

Actual error: $|0.825 - 0.852| = 0.027$

$f'(0.7) = \cos(0.7) = 0.7648$

Actual error: $|0.7648 - 0.796| = 0.0312$

Exercise Set 4.1

4. The data in Exercise 2 were taken from the following functions. Compute the actual errors in Exercise 2, and find error bounds using the error formulas.

a. $f(x) = 2\cos 2x - x$ **b.** $f(x) = x^2 \ln x + 1$

5. Use the most accurate three-point formula to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

b.

x	$f(x)$	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

c.

x	$f(x)$	$f'(x)$
2.9	−4.827866	
3.0	−4.240058	
3.1	−3.496909	
3.2	−2.596792	

d.

x	$f(x)$	$f'(x)$
2.0	3.6887983	
2.1	3.6905701	
2.2	3.6688192	
2.3	3.6245909	

Exercise Set 4.1

7. The data in Exercise 5 were taken from the following functions. Compute the actual errors in Exercise 5, ~~and find error bounds using the error formulas.~~

a. $f(x) = e^{2x}$ **b.** $f(x) = x \ln x$

c. $f(x) = x \cos x - x^2 \sin x$ **d.** $f(x) = 2(\ln x)^2 + 3 \sin x$

a) $f'(x) = 2e^{2x}$

Actual errors:

$f'(1.1) = 2e^{2(1.1)} = 18.050027$	17.769705	$\lvert 18.050027 - 17.769705 \rvert = 0.280322$
$f'(1.2) = 2e^{2(1.2)} = 22.046353$	22.193635	$\lvert 22.046353 - 22.193635 \rvert = 0.147282$
$f'(1.3) = 2e^{2(1.3)} = 26.927476$	27.10735	$\lvert 26.927476 - 27.10735 \rvert = 0.179874$
$f'(1.4) = 2e^{2(1.4)} = 32.88929$	32.51085	$\lvert 32.88929 - 32.51085 \rvert = 0.37844$

Ex 4.1 (9)

5-point method

x	$f(x)$	$f'(x)$
2.1	-1.709847	
2.2	-1.373823	
2.3	-1.119214	
2.4	-0.9160143	
2.5	-0.7470223	
2.6	-0.6015966	

$2.5 - 0.1 = 2.4 = x_0 + h$

$2.5 = x_0$

Midpoint

$$f'(2.3) = \frac{1}{12(0.1)}\left[-1.709847 - 8(-1.373823) + 8(-0.9160143) - (-0.7470223)\right]$$

$$f'(2.4) = \frac{1}{12(0.1)}\left[-1.373823 - 8(-1.119214) + 8(-0.7470223) - (-0.6015966)\right]$$

Backward

$$f'(2.5) = \frac{1}{12(-0.1)}$$

5point endpoint

$$\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\right]$$

Backward $(h<0)$

$$f'(2.5) = \frac{1}{12(-0.1)}\left[-25(-0.7470223) + 48(-0.9160143) - 36(-1.119214) + 16(-1.373823) - 3(-1.709847)\right]$$

$$f'(2.6) = \frac{1}{12(-0.1)}\left[-25(-0.6015966) + 48(-0.7470223) - 36(-0.9160143) + 16(-1.119214) - 3(-1.373823)\right]$$