3.5
CUBIC SPLINE INTERPOLATION

Cubic Spline Interpolation[1]

Cubic spline

Natural   clamp

The previous sections concerned the approximation of arbitrary functions on closed intervals using a single polynomial. However, high-degree polynomials can oscillate erratically; that is, a minor fluctuation over a small portion of the interval can induce large fluctuations over the entire range. We will see a good example of this in Figure 3.14 at the end of this section.

An alternative approach is to divide the approximation interval into a collection of subintervals and construct a (generally) different approximating polynomial on each subinterval. This is called **piecewise-polynomial approximation**.

Piecewise-Polynomial Approximation

The simplest piecewise-polynomial approximation is **piecewise-linear** interpolation, which consists of joining a set of data points

$$\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$$

by a series of straight lines, as shown in Figure 3.7.

Figure 3.7

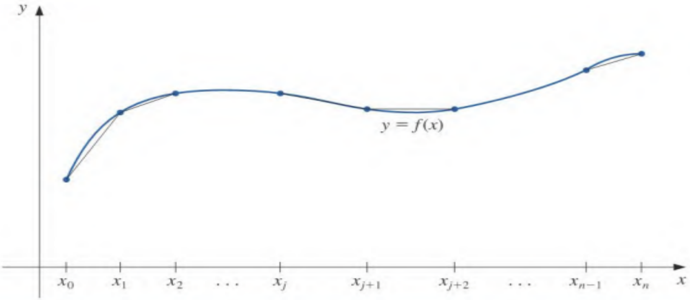
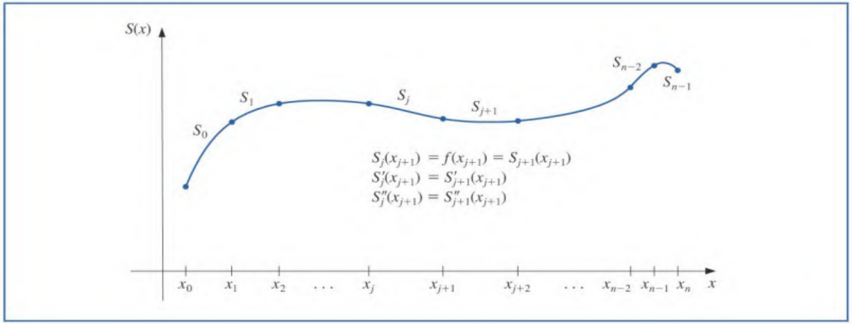


Figure 3.8



Definition 3.10 Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a **cubic spline interpolant** S for f is a function that satisfies the following conditions:

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$;

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$;

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$; (*Implied by* **(b)**.)

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(f) One of the following sets of boundary conditions is satisfied:

(i) $S''(x_0) = S''(x_n) = 0$ (**natural** *(or free)* **boundary**);

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped boundary**).

Natural Splines

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix},$$

and **b** and **x** are the vectors

$$\mathbf{b} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

For natural

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0)$$

$$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 + 2c_1)$$

$$d_0 = \frac{1}{3h_0}(c_1 - c_0)$$

$$d_1 = \frac{1}{3h_1}(c_2 - c_1)$$

for x_0, x_1

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2×2

$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for x_0, x_1, x_2

$$A = \begin{bmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ 0 \end{bmatrix}$$

EXERCISE SET 3.5

EXERCISE SET 3.5

1. Determine the natural cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$.
2. Determine the clamped cubic spline s that interpolates the data $f(0) = 0$, $f(1) = 1$ and $f(2) = 2$ and satisfies $s'(0) = s'(2) = 1$.
3. Construct the natural cubic spline for the following data.

a.

x	$f(x)$
8.3	17.56492
8.6	18.50515

b.

x	$f(x)$
0.8	0.22363362
1.0	0.65809197

c.

x	$f(x)$
−0.5	−0.0247500
−0.25	0.3349375
0	1.1010000

d.

x	$f(x)$
0.1	−0.62049958
0.2	−0.28398668
0.3	0.00660095
0.4	0.24842440

4. Construct the natural cubic spline for the following data.

a.

x	$f(x)$
0	1.00000
0.5	2.71828

b.

x	$f(x)$
−0, 25	1.33203
0.25	0.800781

c.

x	$f(x)$
0.1	−0.29004996
0.2	−0.56079734
0.3	−0.81401972

d.

x	$f(x)$
−1	0.86199480
−0.5	0.95802009
0	1.0986123
0.5	1.2943767

5. The data in Exercise 3 were generated using the following functions. Use the cubic splines constructed in Exercise 3 for the given value of x to approximate $f(x)$ and $f'(x)$ and calculate the actual error.

a. $f(x) = x \ln x$; approximate $f(8.4)$ and $f'(8.4)$.

b. $f(x) = \sin(e^x - 2)$; approximate $f(0.9)$ and $f'(0.9)$.

c. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$; approximate $f(-\frac{1}{3})$ and $f'(-\frac{1}{3})$.

d. $f(x) = x\cos x - 2x^2 + 3x - 1$; approximate $f(0.25)$ and $f'(0.25)$.

Ex: 3.5

Natural spline

3a

x	$f(x)$
$x_0 = 8.3$	$17.56492 = a_0$
$x_1 = 8.6$	$18.50515 = a_1$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$c = A^{-1} b = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} c_0 \\ c_1 \end{array}$$

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0)$$

$$d_0 = \boxed{\frac{1}{3h_0}}(c_1 - c_0)$$

$$h_0 = x_1 - x_0 = 8.6 - 8.3 = 0.3$$

$$b_0 = \frac{1}{0.3}(18.50515 - 17.56492) - \cancel{\frac{0.3}{3}(0)} = 3.1341$$

$$d_0 = \frac{1}{3(0.3)}(0) = 0$$

$$s(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 \quad [x_0, x_1]$$

$$= 17.5692 + 3.1341(x - 8.3) + 0 + 0 \quad [8.3, 8.6]$$

$$= 17.5692 + 3.1341(x - 8.3) \quad [8.3, 8.6]$$

3c

x	$f(x)$
$x_0=-0.5$	$-0.02475=a_0$
$x_1=-0.25$	$0.3344375=a_1$
$x_2=0$	$1.101=a_2$

3x3 natural spline

$$h_0=x_1-x_0=-0.25+0.5=0.25$$
$$h_1=x_2-x_1=0+0.25=0.25$$

$$A^{-1}=\begin{bmatrix}1&0&0\\-0.25&1&-0.25\\0&0&1\end{bmatrix}$$

$$A=\begin{bmatrix}1&0&0\\h_0&2(h_0+h_1)&h_1\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0.25&2(0.25+0.25)&0.25\\0&0&1\end{bmatrix}$$

$$A=\begin{bmatrix}1&0&0\\0.25&1&0.25\\0&0&1\end{bmatrix}$$

$$b=\begin{bmatrix}0\\\frac{3}{h_1}(a_2-a_1)-\frac{3}{h_0}(a_1-a_0)\\0\end{bmatrix}=\begin{bmatrix}0\\\frac{3}{0.25}(1.101-0.3349375)-\frac{3}{0.25}(0.3349375+0.02475)\\0\end{bmatrix}=\begin{bmatrix}0\\4.8765\\0\end{bmatrix}$$

$$A^{-1}b=\begin{bmatrix}1&0&0\\-0.25&1&-0.25\\0&0&1\end{bmatrix}\begin{bmatrix}0\\4.8765\\0\end{bmatrix}=\begin{bmatrix}0\\4.8765\\0\end{bmatrix}\begin{matrix}=c_0\\=c_1\\=c_2\end{matrix}$$

$$b_0=\frac{1}{h_0}(a_1-a_0)-\frac{h_0}{3}(c_1+2c_0)=\frac{1}{0.25}(0.3349375+0.02475)-\frac{0.25}{3}(4.8765+2(0))=1.032375$$

$$b_1=\frac{1}{h_1}(a_2-a_1)-\frac{h_1}{3}(c_2+2c_1)=\frac{1}{0.25}(1.101-0.3349375)-\frac{0.25}{3}(0+2(4.8765))=2.2515$$

$$d_0=\frac{1}{3h_0}(c_1-c_0)=\frac{1}{3(0.25)}(4.8765-0)=6.502$$

$$d_1=\frac{1}{3h_1}(c_2-c_1)=\frac{1}{3(0.25)}(0-4.8765)=-6.502$$

$$S(x)=\begin{cases}a_0+b_0(x-x_0)+c_0(x-x_0)^2+d_0(x-x_0)^3 & [x_0,x_1]\\ a_1+b_1(x-x_1)+c_1(x-x_1)^2+d_1(x-x_1)^3 & [x_1,x_2]\end{cases}$$

$$=\begin{cases}-0.02475+1.032375(x+0.5)+0+6.502(x+0.5)^3 & [-0.5,-0.25]\\ 0.3349375+2.2515(x+0.25)+4.8765(x+0.25)^2-6.502(x+0.25)^3 & [-0.25,0]\end{cases}$$

Clamped Splines

determine the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

For Clamped

for x_0, x_1

$$A = \begin{bmatrix} 2h_0 & h_0 \\ h_0 & 2h_0 \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ 3f'(b) - \frac{3}{h_0}(a_1 - a_0) \end{bmatrix}$$

for x_0, x_1, x_2

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & h_1 & 2h_1 \end{bmatrix}$$

$$b = \begin{bmatrix} \boxed{\frac{3}{h_0}(a_1 - a_0)} - 3f'(a) \\ \boxed{\frac{3}{h_1}(a_2 - a_1)} - \boxed{\frac{3}{h_0}(a_1 - a_0)} \\ 3f'(b) - \boxed{\frac{3}{h_1}(a_2 - a_1)} \end{bmatrix}$$

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c. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$; approximate $f(-\frac{1}{3})$ and $f'(-\frac{1}{3})$.

d. $f(x) = x \cos x - 2x^2 + 3x - 1$; approximate $f(0.25)$ and $f'(0.25)$.

7. Construct the clamped cubic spline using the data of Exercise 3 and the fact that

a. $f'(8.3) = 1.116256$ and $f'(8.6) = 1.151762$.

b. $f'(0.8) = 2.1691753$ and $f'(1.0) = 2.0466965$.

c. $f'(-0.5) = 0.7510000$ and $f'(0) = 4.0020000$.

d. $f'(0.1) = 3.58502082$ and $f'(0.4) = 2.16529366$.

8. Construct the clamped cubic spline using the data of Exercise 3 and the fact that

a. $f'(0) = 2$ and $f'(0.5) = 5.43656$.

b. $f'(-0.25) = 0.437500$ and $f'(0.25) = -0.625000$.

c. $f'(0.1) = -2.8004996$ and $f'(0) = -2.9734038$.

d. $f'(-1) = 0.15536240$ and $f'(0.5) = 0.45186276$.

11. A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x < 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find b, c, and d.

12. A natural cubic spline S is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \le x < 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

If S interpolates the data $(1, 1)$, $(2, 1)$, and $(3, 0)$, find B, D, b, and d.

13. A clamped cubic spline s for a function f is defined on $[1, 3]$ by

$$s(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \le x < 2, \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

Given $f'(1) = f'(3)$, find a, b, c, and d.

14. A clamped cubic spline s for a function f is defined by

$$s(x) = \begin{cases} s_0(x) = 1 + Bx + 2x^2 - 2x^3, & \text{if } 0 \le x < 1, \\ s_1(x) = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find $f'(0)$ and $f'(2)$.

7a

x	$f(x)$	$f'(x)$
$x_0=8.3$	$17.56492=a_0$	$3.116256=f'(a)$
$x_1=8.6$	$18.50515=a_1$	$3.151762=f'(b)$

$$A=\begin{bmatrix}2h_0 & h_0\\ h_0 & 2h_0\end{bmatrix}=\begin{bmatrix}2(0.3) & 0.3\\ 0.3 & 2(0.3)\end{bmatrix}=\begin{bmatrix}0.6 & 0.3\\ 0.3 & 0.6\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}\frac{20}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{20}{9}\end{bmatrix}$$

In fraction because of recurring decimal

$$b=\begin{bmatrix}\frac{3}{h_0}(a_1-a_0)-3f'(a)\\ 3f'(b)-\frac{3}{h_0}(a_1-a_0)\end{bmatrix}=\begin{bmatrix}\frac{3}{0.3}(18.50515-17.56492)-3(3.116256)\\ 3(3.151762)-\frac{3}{0.3}(18.50515-17.56492)\end{bmatrix}=\begin{bmatrix}0.053532\\ 0.052986\end{bmatrix}$$

$$C=A^{-1}b=\begin{bmatrix}\frac{20}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{20}{9}\end{bmatrix}\begin{bmatrix}0.053532\\ 0.052986\end{bmatrix}=\begin{bmatrix}0.0600867\\ 0.0582667\end{bmatrix}\begin{array}{l}=c_0\\ =c_1\end{array}$$

$$h_0=x_1-x_0=8.6-8.3=0.3$$

$$b_0=\frac{1}{h_0}(a_1-a_0)-\frac{h_0}{3}(c_1+2c_0)$$

$$b_0=\frac{1}{0.3}(18.50515-17.56492)-\frac{0.3}{3}(0.0582667-2(0.0600867))=3.116256$$

$$d_0=\frac{1}{3h_0}(c_1-c_0)$$

$$d_0=\frac{1}{3(0.3)}(0.0582667-0.0600867)=-0.00202222$$

$$s(x)=a_0+b_0(x-x_0)+c_0(x-x_0)^2+d_0(x-x_0)^3\ [x_0,x_1]$$

$$=17.5692+3.116256(x-8.3)+0.0600867(x-8.3)^2-0.00202222(x-8.3)^3\ [8.3,8.6]$$

7c

3×3 Clamped spline

x	$f(x)$	$f'(x)$
$x_0=-0.5$	$-0.02475=a_0$	$f'(-0.5)=0.751=f'(a)$
$x_1=-0.25$	$0.3344375=a_1$	$f'(-0.25)=4.002=f'(b)$
$x_2=0$	$1.101=a_2$	

$h_0=x_1-x_0=-0.25+0.5=0.25$

$h_1=x_2-x_1=0+0.25=0.25$

$$A=\begin{bmatrix}2h_0 & h_0 & 0\\ h_0 & 2(h_0+h_1) & h_1\\ 0 & h_1 & 2h_0\end{bmatrix}$$

$$=\begin{bmatrix}2(0.25) & 0.25 & 0\\ 0.25 & 2(0.25+0.25) & 0.25\\ 0 & 0.25 & 2(0.25)\end{bmatrix}$$

$$A=\begin{bmatrix}0.5 & 0.25 & 0\\ 0.25 & 1 & 0.25\\ 0 & 0.25 & 0.5\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}7/3 & -2/3 & 1/3\\ -2/3 & 4/3 & -2/3\\ 1/3 & -2/3 & 7/3\end{bmatrix}$$

$$b=\begin{bmatrix}\frac{3}{h_0}(a_1-a_0)-3f'(a)\\ \frac{3}{h_1}(a_2-a_1)-\frac{3}{h_0}(a_1-a_0)\\ 3f'(b)-\frac{3}{h_1}(a_2-a_1)\end{bmatrix}=\begin{bmatrix}\frac{3}{0.25}(0.3349375+0.02475)-3(0.751)\\ \frac{3}{0.25}(1.101-0.3349375)-\frac{3}{0.25}(0.3349375+0.02475)\\ 3(4.002)-\frac{3}{0.25}(1.101-0.3349375)\end{bmatrix}=\begin{bmatrix}2.06325\\ 4.8765\\ 2.81325\end{bmatrix}$$

$$A^{-1}b=\begin{bmatrix}7/3 & -2/3 & 1/3\\ -2/3 & 4/3 & -2/3\\ 1/3 & -2/3 & 7/3\end{bmatrix}\begin{bmatrix}2.06325\\ 4.8765\\ 2.81325\end{bmatrix}=\begin{array}{c|c}\begin{bmatrix}2.501\\ 3.251\\ 4.001\end{bmatrix} & \begin{matrix}c_0\\ c_1\\ c_2\end{matrix}\end{array}$$

$$b_0=\frac{1}{h_0}(a_1-a_0)-\frac{h_0}{3}(c_1+2c_0)=\frac{1}{0.25}(0.3349375+0.02475)-\frac{0.25}{3}(3.251-2(2.501))=0.751$$

$$b_1=\frac{1}{h_1}(a_2-a_1)-\frac{h_1}{3}(c_2+2c_1)=\frac{1}{0.25}(1.101-0.3349375)-\frac{0.25}{3}(4.001-2(3.251))=2.189$$

$$d_0=\frac{1}{3h_0}(c_1-c_0)=\frac{1}{3(0.25)}(3.251-2.501)=1$$

$$d_1=\frac{1}{3h_1}(c_2-c_1)=\frac{1}{3(0.25)}(4.001-3.251)=1$$

$$S(x)=\begin{cases}a_0+b_0(x-x_0)+c_0(x-x_0)^2+d_0(x-x_0)^3 & [x_0,x_1]\\ a_1+b_1(x-x_1)+c_1(x-x_1)^2+d_1(x-x_1)^3 & [x_1,x_2]\end{cases}$$

$$=\begin{cases}-0.02475+0.751(x+0.5)+2.501(x+0.5)^2+(x+0.5)^3 & [-0.5,-0.25]\\ 0.3349375+2.189(x+0.25)+3.251(x+0.25)^2+(x+0.25)^3 & [-0.25,0]\end{cases}$$

7c

$$b = \begin{bmatrix} \frac{3}{h_0}(a_1-a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ 3f'(b) - \frac{3}{h_1}(a_2-a_1) \end{bmatrix} = \begin{bmatrix} \frac{3}{0.25}(0.3349375+0.02475) - 3(0.751) \\ \frac{3}{0.25}(1.101-0.3349375) - \frac{3}{0.25}(0.3349375+0.02475) \\ 3(4.002) - \frac{3}{0.25}(1.101-0.3349375) \end{bmatrix}$$

$$b_0 = \frac{1}{h_0}(a_1-a_0) - \frac{h_0}{3}(c_1+2c_0) = \frac{1}{0.25}(0.3349375+0.02475) - \frac{0.25}{3}(3.251+2(2.501))$$

$$= 0.751$$

$$b_1 = \frac{1}{h_1}(a_2-a_1) - \frac{h_1}{3}(c_2+2c_1) = \frac{1}{0.25}(1.101-0.3349375) - \frac{0.25}{3}(4.001+2(3.251))$$

$$= 2.189$$

$$d_0 = \frac{1}{3h_0}(c_1-c_0) = \frac{1}{3(0.25)}[3.251-2.501] = 1$$

$$d_1 = \frac{1}{3h_1}(c_2-c_1) = \frac{1}{3(0.25)}[4.001-3.251] = 1$$

$$a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)$$

$$S(x) = \begin{cases} -0.02475 + 0.751(x+0.5) + 2.501(x+0.5)^2 + (x+0.5)^3, & [-0.5,-0.25] \ ① \\ 0.3349375 + 2.189(\underset{-0.1}{x}+0.25) + 3.251(\underset{-0.1}{x}+0.25)^2 + (\underset{-0.1}{x}+0.25)^3, & [\underline{-0.25, 0}] \ ② \end{cases}$$

-0.1 ②

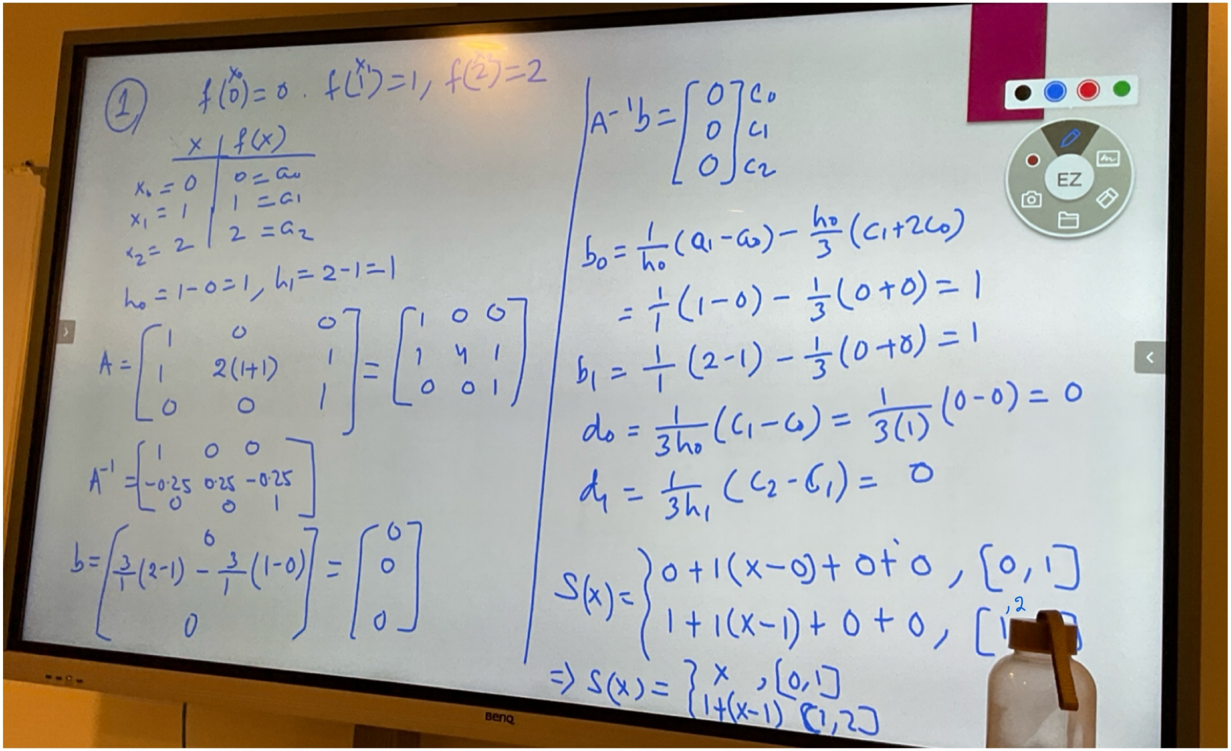
$[0, 0.25]$ ①

$[0.25, 0.5]$ ②

$x = 0.1$ ①

$x = 0.3$ ②

(1) $f(0)=0$ $\quad f(1)=1$ $\quad f(2)=2$



$\Rightarrow s(x) = x, [0,2]$

Ex° 3.5 Natural spline

3a

x	$f(x)$
8.3	17.56492
8.6	18.50515

$$\left.\begin{aligned} b_0 &= \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0) \\ d_0 &= \frac{1}{3h_0}(c_1 - c_0) \end{aligned}\right\} \text{given}$$

$$h_0 = x_1 - x_0 = 8.6 - 8.3 = 0.3$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$c = A^{-1} b = \begin{array}{cc} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{matrix} c_0 \\ c_1 \end{matrix} \end{array}$$

$$b_0 = \frac{1}{0.3}(18.50515 - 17.56492) - \frac{0.3}{3}(0) = 3.1341$$

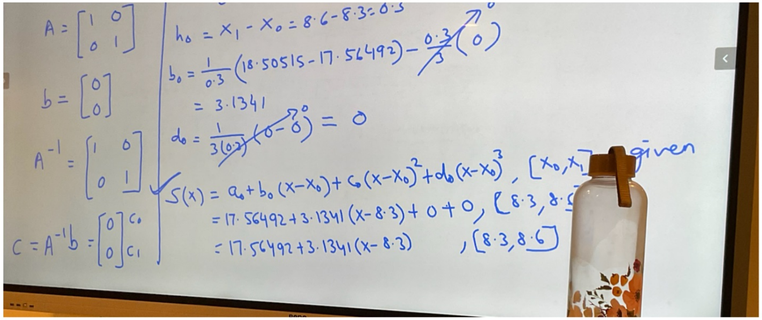
$$d_0 = \frac{1}{3(0.3)}(0) = 0$$

given

$$s(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 \quad [x_0, x_1]$$

$$= 17.5692 + 3.1341(x - 8.3) + 0 + 0 \quad [8.3, 8.6]$$

$$= 17.5692 + 3.1341(x - 8.3) \quad [8.3, 8.6]$$



3c

x	$f(x)$
$x_0=-0.5$	$-0.02475 = a_0$
$x_1=-0.25$	$0.3349375 = a_1$
$x_2=0$	$1.101 = a_2$

3x3 natural spline

$$h_0 = x_1 - x_0 = -0.25+0.5 = 0.25$$

$$h_1 = x_2 - x_1 = 0+0.25 = 0.25$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & -0.25 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 2(0.25+0.25) & 0.25 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0.25 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} \boxed{0} \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \boxed{0} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{0.25}(1.101-0.3349375) - \frac{3}{0.25}(0.3349375+0.02475) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4.8765 \\ 0 \end{bmatrix}$$

$$A^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4.8765 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4.8765 \\ 0 \end{bmatrix} \begin{matrix} = c_0 \\ = c_1 \\ = c_2 \end{matrix}$$

$$b_0 = \frac{1}{h_0}(a_1-a_0) - \frac{h_0}{3}(c_1+2c_0) = \frac{1}{0.25}(0.3349375+0.02475) - \frac{0.25}{3}(4.8765+2(0)) = 1.032375$$

$$b_1 = \frac{1}{h_1}(a_2-a_1) - \frac{h_1}{3}(c_2+2c_1) = \frac{1}{0.25}(1.101-0.3349375) - \frac{0.25}{3}(0+2(4.8765)) = 2.2515$$

$$d_0 = \frac{1}{3h_0}(c_1-c_0)$$

$$d_0 = \frac{1}{3(0.25)}(4.8765-0) = 6.502$$

$$d_1 = \frac{1}{3h_1}(c_2-c_1)$$

$$= \frac{1}{3(0.25)}(0-4.8765) = -6.502$$

x_0, x_1, x_2

$x_0 = -0.5$

$$S(x) = \begin{cases} a_0 + b_0(x-x_0) + \cancel{c_0(x-x_0)^2} + d_0(x-x_0)^3, & [x_0, x_1] \\ a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3, & [x_1, x_2] \end{cases}$$

$x = 0.3$

$$= \begin{cases} -0.02475 + 1.032375(x+0.5) + 0 + 6.502(x+0.5)^3, & [-0.5, -0.25] \\ 0.3349375 + 2.2515(x+0.25) + 4.8765(x+0.25)^2 - 6.502(x+0.25)^3, & [-0.25, 0] \end{cases}$$

Ex 3.5

5a $f(x) = x \ln x$, approx $f(8.4)$, $f'(8.4)$

$$S(x) = 17.56492 + 3.1341(x - 8.3) \qquad [8.3, 8.6]$$

from 3a

approx value $S(8.4) = 17.56492 + 3.1341(8.4 - 8.3) = 17.87833$

Exact value $f(8.4) = 8.4 \ln(8.4) = 17.8771463$

$$A.E = |f(8.4) - S(8.4)| = |17.8771463 - 17.87833|$$
$$= 0.006867$$

$$f'(x) = 1(\ln x) + \cancel{x} \cdot \frac{1}{\cancel{x}} = \ln x + 1$$

$$f'(8.4) = \ln(8.4) + 1 = 3.128232$$

$$s'(x) = 3.1341 \Rightarrow s'(8.4) = 3.1341$$

$$|f'(8.4) - s'(8.4)| = |3.128232 - 3.1341|$$
$$= 0.0058$$

$f(-\frac{1}{3}),\ f'(-\frac{1}{3})$

Ex 3.5

(5c) $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$,

$f(-\frac{1}{3}) = (-\frac{1}{3})^3 + 4.001(-\frac{1}{3})^2 + 4.002(-\frac{1}{3}) + 1.101$

$= 0.174519$

$f'(x) = 3x^2 + 4.001(2x) + 4.002$

$= 3x^2 + 8.002x + 4.002$

$f'(-\frac{1}{3}) = 3(-\frac{1}{3})^2 + 8.002(-\frac{1}{3}) + 4.002 = 1.668$

Ex 3.16

$$S(x) = \begin{cases} -0.02475 + 1.032375(x+\frac{1}{2}) + 6.502(x+\frac{1}{2})^3, & [-0.5, -0.25] \\ 0.3349375 + 2.2515(x+\frac{1}{4}) + 4.8765(x+\frac{1}{4})^2 - 6.502(x+\frac{1}{4})^3, & [-0.25, 0] \end{cases}$$

$-\frac{1}{3} \in [-0.5, -0.25]$

$S(-\frac{1}{3}) = -0.02475 + 1.032375(-\frac{1}{3}+\frac{1}{2}) + 6.502(-\frac{1}{3}+\frac{1}{2})^3 = 0.177414$

$A.E = |f(-\frac{1}{3}) - S(-\frac{1}{3})| = |0.174519 - 0.177414| = 0.0028$

$S'(x) = 1.032375 + 6.502(3)(x+\frac{1}{2})^2$

$S'(-\frac{1}{3}) = 1.032375 + 6.502(3)(-\frac{1}{3}+\frac{1}{2})^2 = 1.57421$

$A.E = |f'(-\frac{1}{3}) - S'(-\frac{1}{3})|$

$= |1.668 - 1.57421|$

$= 0.093$

7c clamped

	x	$f(x)$	$f'(x)$
x_0	-0.5	$-0.02475 = a_0$	$f'(-0.5) = 0.751 = f'(a)$
x_1	-0.25	$0.3349375 = a_1$	$f'(0) = 4.002 = f'(b)$
x_2	0	$1.101 = a_2$	

$h = 0.25$

$h_0 = -0.25 + 0.5 = 0.25$

$h_1 = 0 + 0.25 = 0.25$

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & h_1 & 2h_1 \end{bmatrix} = \begin{bmatrix} 2(0.25) & 0.25 & 0 \\ 0.25 & 2(0.25+0.25) & 0.25 \\ 0 & 0.25 & 2(0.25) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.25 & 1 & 0.25 \\ 0 & 0.25 & 0.5 \end{bmatrix}$$

$C = A^{-1} b$

$\frac{3}{0.25}(0.3349375 - (-0.02475)) - 3(0.751)$

$\frac{3}{0.25}(1.101 - 0.3349375) - \frac{3}{0.25}($

$$b = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ 3f'(b) - \frac{3}{h_1}(a_2 - a_1) \end{bmatrix} = \begin{bmatrix} \frac{3}{0.25}(0.3349375 + 0.02475) - 3(0.751) \\ \frac{3}{0.25}(1.101 - 0.3349375) - \frac{3}{0.3}(0.3349375 + 0.02475) \\ 3(4.002) - \frac{3}{0.3}(1.101 - 0.3349375) \end{bmatrix}$$

$$= \begin{bmatrix} 2.06325 \\ \cancel{5.595875} \\ \cancel{4.345375} \end{bmatrix} \begin{matrix} b_0 \\ 4.87650\ b_1 \\ 2.81325\ b_2 \end{matrix}$$

$$A^{-1} = \begin{bmatrix} 7/3 & -2/3 & 1/3 \\ -2/3 & 4/3 & -2/3 \\ 1/3 & -2/3 & 7/3 \end{bmatrix},$$

$$A^{-1}b = \begin{bmatrix} 7/3 & -2/3 & 1/3 \\ -2/3 & 4/3 & -2/3 \\ 1/3 & -2/3 & 7/3 \end{bmatrix} \begin{bmatrix} 2.06325 \\ 5.595875 \\ 4.345375 \end{bmatrix} = \begin{bmatrix} 2.532125 \\ 3.18875 \\ 7.096375 \end{bmatrix}$$

$$\begin{bmatrix} 2.501 \Rightarrow c_1 \\ 3.251 \Rightarrow c_2 \\ 4.001 \Rightarrow c_3 \end{bmatrix}$$

7c

$$b = \begin{bmatrix} \frac{3}{h_0}(a_1-a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ 3f'(b) - \frac{3}{h_1}(a_2-a_1) \end{bmatrix} = \begin{bmatrix} \frac{3}{0.25}(0.3349375+0.02475) - 3(0.751) \\ \frac{3}{0.25}(1.101-0.3349375) - \frac{3}{0.25}(0.3349375+0.02475) \\ 3(4.002) - \frac{3}{0.25}(1.101-0.3349375) \end{bmatrix}$$

$$b_0 = \frac{1}{h_0}(a_1-a_0) - \frac{h_0}{3}(c_1+2c_0) = \frac{1}{0.25}(0.3349375+0.02475) - \frac{0.25}{3}(3.251+2(2.501))$$
$$= 0.751$$

$$b_1 = \frac{1}{h_1}(a_2-a_1) - \frac{h_1}{3}(c_2+2c_1) = \frac{1}{0.25}(1.101-0.3349375) - \frac{0.25}{3}(4.001+2(3.251))$$
$$= 2.189$$

$$d_0 = \frac{1}{3h_0}(c_1-c_0) = \frac{1}{3(0.25)}[3.251-2.501] = 1$$

$$d_1 = \frac{1}{3h_1}(c_2-c_1) = \frac{1}{3(0.25)}[4.001-3.251] = 1$$

$$a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)$$

$$S(x) = \begin{cases} -0.02475 + 0.751(x+0.5) + 2.501(x+0.5)^2 + (x+0.5)^3, & [-0.5, -0.25] \ ① \\ 0.3349375 + 2.189(\underset{-0.1}{x}+0.25) + 3.251(\underset{-0.1}{x}+0.25)^2 + (\underset{-0.1}{x}+0.25)^3, & [-0.25, 0] \ ② \end{cases}$$

-0.1 ②

$[0, 0.25]$ ①

$[0.25, 0.5]$ ②

$x = 0.1$ ①

$x = 0.3$ ②

11. A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x < 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2. \end{cases}$$

Find b, c, and d.

$\boxed{X = 1}$

$S_0'(x) = S_1'(x)$ $\quad 1 + 2(1) - 1^3 = \boxed{2}$ equal

$2 + \cancel{b(1-1)} + \cancel{c(1-1)^2} + \cancel{d(1-1)^3} = \boxed{2}$

$S_0'(x) = 2 - 3x^2 \quad S_0'(1) = 2 - 3(1)^2 = -1$ should be equal so $b = -1$

$S_1'(x) = b + 2c(x-1) + 3d(x-1)^2 \quad S_1'(1) = b + 2c\cancel{(1-1)} + 3d\cancel{(1-1)^2} = b$

$S_0''(x) = -6x \quad S_0''(1) = -6(1) = -6$

$-6 = 2c$

$c = -3$

b)

$S_1''(x) = 2c + 6d(x-1) \quad S_1''(1) = 2c + 6d\cancel{(1-1)} = 2c$

for natural spline $\underset{\text{1st point}}{S_0''(0)} = \underset{\text{last point}}{S_1''(2)}$ $\quad S_0'(a) = S_1'($

$-6(0) = 2c + 6d(2-1)$

$0 = 2c + 6d$

$0 = 2(-3) + 6d$

$0 = -6 + 6d$

$6 = 6d$

$\boxed{d = 1}$

13. A clamped cubic spline s for a function f is defined on $[1, 3]$ by

$3\cancel{x}-3 + 2(x-1)^2 - (x-1)^3$

$$s(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \le x < 2, \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

$b\cancel{x} - 2b$

Given $f'(1) = f'(3)$, find a, b, c, and d.

$a = 4$

$S_0(2) = 3(2-1) + 2(2-1)^2 - (2-1)^3 = 4$

$S_1(2) = a + b\cancel{(2-2)} + c\cancel{(2-2)^2} + \cancel{d(2-2)^3} = a$

$S_0'(x) = 3 + 2(2)(x-1) - 3(x-1)^2$ $S_0'(2) = 3 + 4(2-1) - 3(2-1)^2 = 4$

4

$b = 4$

$S_1'(x) = b + 2c(x-2) + 3d(x-2)^2$ $S_1'(2) = b + 2c\cancel{(2-2)} + \cancel{3d(2-2)^2} = b$

$\frac{2c}{2} = \frac{-2}{2}$

$S_0''(x) = 4 - 3(2)(x-1)$ $S_0''(2) = 4 - 6(2-1) = -2$

$c = -1$

$S_1''(x) = 2c + 3d(2)(x-2)$ $S_1''(2) = 2c + \cancel{6d(2-2)} = 2c$

$S_0'(1) = S_1'(3)$

$3 + \cancel{4(1-1)} - \cancel{3(1-1)^2} = b + 2c(3-2) + 3d(3-2)^2$

$3 = 4 + 2(-1) + 3d$

$3 = 2 + 3d$

$-2 \quad -2$

$\frac{1}{3} = \frac{3d}{3}$ $d = \frac{1}{3}$