3.4

Hermite Interpolation

Hermite Polynomials Using Divided Differences

There is an alternative method for generating Hermite approximations that has as its basis the Newton interpolatory divided-difference formula (3.10) at $x_0, x_1, \ldots, x_n$; that is,

$$\star \quad P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

formula will be given

The alternative method uses the connection between the nth divided difference and the nth derivative of f, as outlined in Theorem 3.6 in Section 3.3.

Suppose that the distinct numbers $x_0, x_1, \ldots, x_n$ are given together with the values of f and f' at these numbers. Define a new sequence $z_0, z_1, \ldots, z_{2n+1}$ by

$$z_{2i} = z_{2i+1} = x_i, \quad \text{for each } i = 0, 1, \ldots, n,$$

and construct the divided difference table in the form of Table 3.9 that uses $z_0, z_1, \ldots, z_{2n+1}$.

Since $z_{2i} = z_{2i+1} = x_i$ for each i, we cannot define $f[z_{2i}, z_{2i+1}]$ by the divided-difference formula. However, if we assume, based on Theorem 3.6, that the reasonable substitution in this situation is $f[z_{2i}, z_{2i+1}] = f'(z_{2i}) = f'(x_i)$, we can use the entries

$$f'(x_0), f'(x_1), \ldots, f'(x_n)$$

in place of the undefined first divided differences

$$f[z_0, z_1], f[z_2, z_3], \ldots, f[z_{2n}, z_{2n+1}].$$

The remaining divided differences are produced as usual, and the appropriate divided differences are employed in Newton's interpolatory divided-difference formula. Table 3.16 shows the entries that are used for the first three divided-difference columns when determining the Hermite polynomial $H_5(x)$ for x_0, x_1, and x_2. The remaining entries are generated in the same manner as in Table 3.9. The Hermite polynomial is then given by

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \ldots, z_k](x - z_0)(x - z_1) \cdots (x - z_{k-1}).$$

A proof of this fact can be found in [Pow], p. 56.

Table 3.16

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \dfrac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0}$
		$f[z_1, z_2] = \dfrac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \dfrac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \dfrac{f[z_3, z_4] - f[z_2, z_3]}{z_4 - z_2}$
		$f[z_3, z_4] = \dfrac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \dfrac{f[z_4, z_5] - f[z_3, z_4]}{z_5 - z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

Example 2 Use the data given in Example 1 and the divided difference method to determine the Hermite polynomial approximation at $x = 1.5$.

Solution The underlined entries in the first three columns of Table 3.17 are the data given in Example 1. The remaining entries in this table are generated by the standard divided-difference formula (3.9).

For example, for the second entry in the third column we use the second 1.3 entry in the second column and the first 1.6 entry in that column to obtain

$$\frac{0.4554022 - 0.6200860}{1.6 - 1.3} = -0.5489460.$$

For the first entry in the fourth column, we use the first 1.3 entry in the third column and the first 1.6 entry in that column to obtain

$$\frac{-0.5489460 - (-0.5220232)}{1.6 - 1.3} = -0.0897427.$$

The value of the Hermite polynomial at 1.5 is

$$\begin{aligned}
H_5(1.5) = {} & f[1.3] + f'(1.3)(1.5 - 1.3) + f[1.3, 1.3, 1.6](1.5 - 1.3)^2 \\
& + f[1.3, 1.3, 1.6, 1.6](1.5 - 1.3)^2(1.5 - 1.6) \\
& + f[1.3, 1.3, 1.6, 1.6, 1.9](1.5 - 1.3)^2(1.5 - 1.6)^2 \\
& + f[1.3, 1.3, 1.6, 1.6, 1.9, 1.9](1.5 - 1.3)^2(1.5 - 1.6)^2(1.5 - 1.9) \\
= {} & 0.6200860 + (-0.5220232)(0.2) + (-0.0897427)(0.2)^2 \\
& + 0.0663657(0.2)^2(-0.1) + 0.0026663(0.2)^2(-0.1)^2 \\
& + (-0.0027738)(0.2)^2(-0.1)^2(-0.4) \\
= {} & 0.5118277.
\end{aligned}$$

■

Table 3.17

1.3	0.6200860					
		−0.5220232				
1.3	0.6200860		−0.0897427			
		−0.5489460		0.0663657		
1.6	0.4554022		−0.0698330		0.0026663	
		−0.5698959		0.0679655		−0.0027738
1.6	0.4554022		−0.0290537		0.0010020	
		−0.5786120		0.0685667		
1.9	0.2818186		−0.0084837			
		−0.5811571				
1.9	0.2818186					

EXERCISE SET 3.4

1. Use Theorem 3.9 or Algorithm 3.3 to construct an approximating polynomial for the following data.

a.

x	$f(x)$	$f'(x)$
8.3	17.56492	3.116256
8.6	18.50515	3.151762

b.

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1.0	0.65809197	2.0466965

c.

x	$f(x)$	$f'(x)$
−0.5	−0.0247500	0.7510000
−0.25	0.3349375	2.1890000
0	1.1010000	4.0020000

d.

x	$f(x)$	$f'(x)$
0.1	−0.62049958	3.58502082
0.2	−0.28398668	3.14033271
0.3	0.00660095	2.66668043
0.4	0.24842440	2.16529366

* more than 3 points
won't come
(max 3 points)

2. Use Theorem 3.9 or Algorithm 3.3 to construct an approximating polynomial for the following data.

a.

x	$f(x)$	$f'(x)$
0	1.00000	2.00000
0.5	2.71828	5.43656

b.

x	$f(x)$	$f'(x)$
−0.25	1.33203	0.437500
0.25	0.800781	−0.625000

c.

x	$f(x)$	$f'(x)$
0.1	−0.29004996	−2.8019975
0.2	−0.56079734	−2.6159201
0.3	−0.81401972	−2.9734038

d.

x	$f(x)$	$f'(x)$
−1	0.86199480	0.15536240
−0.5	0.95802009	0.23269654
0	1.0986123	0.33333333
0.5	1.2943767	0.45186776

1. Use Theorem 3.9 or Algorithm 3.3 to construct an approximating polynomial for the following data.

a.

x	$f(x)$	$f'(x)$
8.3	17.56492	3.116256
8.6	18.50515	3.151762

$f'(x_0)$ $f'(8.3)$

b.

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1.0	0.65809197	2.0466965

1) a) Hermite interpolation

copy from question

$z_0 = x_0 = 8.3$ $f(x_0) = f(z_0) = 17.56492$

$z_1 = x_0 = 8.3$ $f(x_0) = f(z_1) = 17.56492$

$z_2 = x_1 = 8.6$ $f(x_1) = f(z_2) = 18.50515$

$z_3 = x_1 = 8.6$ $f(x_1) = f(z_3) = 18.50515$

$3.116256 = f(z_0, z_1)$

$3.1341 = f(z_1, z_2)$ calculate

$3.151762 = f(z_2, z_3)$

copy from question

$f(z_0, z_1, z_2) = 0.05948$

$f(z_1, z_2, z_3) = 0.0588733$

$f(z_0, z_1, z_2, z_3) = -0.002022$

$$f(z_1, z_2) = \frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{18.50515 - 17.56492}{8.6 - 8.3} = 3.1341$$

$$f(z_0, z_1, z_2) = \frac{f(z_1, z_2) - f(z_0, z_1)}{z_2 - z_0} = \frac{3.1341 - 3.116256}{8.6 - 8.3} = 0.05948$$

$$f(z_1, z_2, z_3) = \frac{f(z_2, z_3) - f(z_1, z_2)}{z_3 - z_1} = \frac{3.151762 - 3.1341}{8.6 - 8.3} = 0.0588733$$

$$f(z_0, z_1, z_2, z_3) = \frac{f(z_1, z_2, z_3) - f(z_0, z_1, z_2)}{f(z_3 - z_1)} = \frac{0.0588733 - 0.05948}{8.6 - 8.6} = -0.002022$$

polynomial: $H_{2n+1}(x) = f(z_0) + \sum_{k=1}^{2n+1} f(z_0, \dots z_k)(x - z_0)(x - z_1) \dots (x - z_{k-1})$

$n = 1$

$2(1)+1 = 3$

2 terms

$$H_3(x) = f(z_0) + \sum_{k=1}^{3} f(z_0, z_1, \dots z_k)(x - z_0)(x - z_1) \dots (x - z_{k-1})$$

$$= f(z_0) + f(z_0, z_1)(x - z_0) + f(z_0, z_1, z_2)(x - z_0)(x - z_1) + f(z_0, z_1, z_2, z_3)(x - z_0)(x - z_1)(x - z_2)$$

$$= 17.56492 + 3.116256(x - 8.3) + 0.05948(x - 8.3)(x - 8.3) + (-0.002022)(x - 8.3)(x - 8.3)(x - 8.6)$$

c.

x	$f(x)$	$f'(x)$
-0.5	-0.0247500	0.7510000
-0.25	0.3349375	2.1890000
0	1.1010000	4.0020000

Continuation to question 1

$x_0 = z_0 = -0.5$ $\quad f(z_0) = -0.02475$

$0.751 = f(z_0, z_1)$

$x_0 = z_1 = -0.5$ $\quad f(z_1) = -0.02475$

$f(z_0, z_1, z_2) = 2.751$

$1.43875 = f(z_1, z_2)$

$f(z_0, z_1, z_2, z_3) = 1$

$x_1 = z_2 = -0.25$ $\quad f(z_2) = 0.3349375$

$f(z_1, z_2, z_3) = 3.001$

$2.189 = f(z_2, z_3)$

$f(z_1, z_2, z_3, z_4) = 1$

$x_1 = z_3 = -0.25$ $\quad f(z_3) = 0.3349375$

$f(z_2, z_3, z_4) = 3.501$

$3.06425 = f(z_3, z_4)$

$f(z_2, z_3, z_4, z_5) = 1$

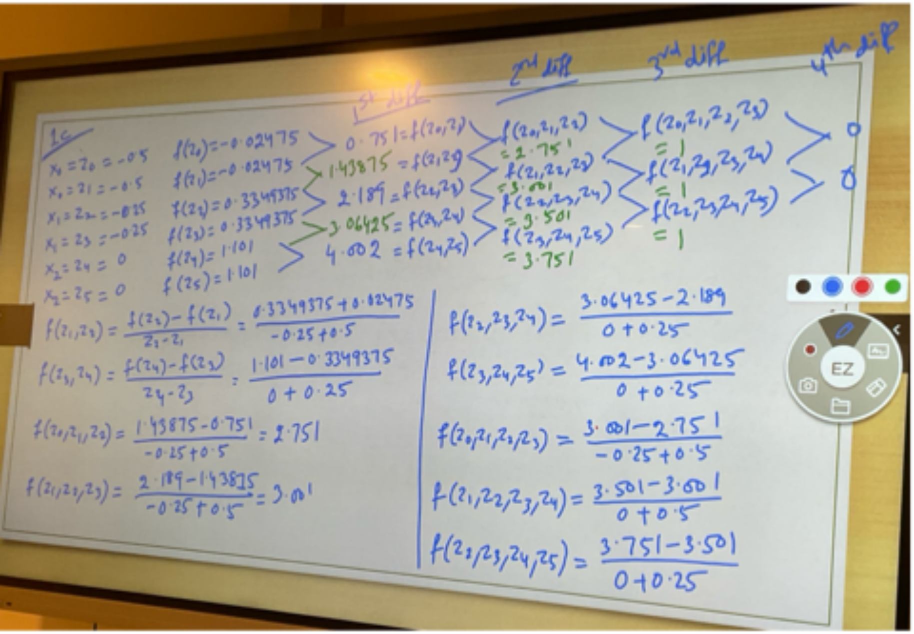
$x_2 = z_4 = 0$ $\quad f(z_4) = 1.101$

$f(z_3, z_4, z_5) = 3.751$

$4.002 = f(z_4, z_5)$

$x_2 = z_5 = 0$ $\quad f(z_5) = 1.101$

* Once the values are the same, stop as the next will always be 0



$$H_5(x) = -0.02475 + 0.751(x - z_0) + 2.751(x - z_0)(x - z_1) + 1(x - z_0)(x - z_1)(x - z_2)$$

$$= -0.02475 + 0.751(x - 0.5) + 2.751(x + 0.5)^2 + (x + 0.5)^2(x + 0.25)$$

* If the rest isn't 0

$$H_5(x) = f(z_0) + f(z_0, z_1)(x - z_0) + f(z_0, z_1, z_2)(x - z_0)(x - z_1) + f(z_0, z_1, z_2, z_3)(x - z_0)(x - z_1)(x - z_2)$$
$$+ f(z_0, z_1, z_2, z_3, z_4)(x - z_0)(x - z_1)(x - z_2)(x - z_3)$$
$$+ f(z_0, z_1, z_2, z_3, z_4, z_5)(x - z_0)(x - z_1)(x - z_2)(x - z_3)(x - z_4)$$

2. Use Theorem 3.9 or Algorithm 3.3 to construct an approximating polynomial for the following data.

c.

x	$f(x)$	$f'(x)$
0.1	−0.29004996	−2.8019975
0.2	−0.56079734	−2.6159201
0.3	−0.81401972	−2.9734038

Ex 3.4 Q2C

1st diff, 2nd diff, 3rd diff

$x_0 = z_0 = 0.1$ $\quad f(z_0) = -0.29004996$
$x_0 = z_1 = 0.1$ $\quad f(z_1) = -0.29004996$
$x_1 = z_2 = 0.2$ $\quad f(z_2) = -0.56079734$
$x_1 = z_3 = 0.2$ $\quad f(z_3) = -0.56079734$
$x_2 = z_4 = 0.3$ $\quad f(z_4) = -0.81401972$
$x_2 = z_5 = 0.3$ $\quad f(z_5) = -0.81401972$

1st diff:
$f(z_0,z_1) = -2.8019975$
$f(z_1,z_2) = -2.7074738$
$f(z_2,z_3) = -2.6159201$
$f(z_3,z_4) = -2.5322238$
$f(z_4,z_5) = -2.9734038$

2nd diff:
$f(z_0,z_1,z_2) = 0.945237$
$f(z_1,z_2,z_3) = 0.915537$
$f(z_2,z_3,z_4) = 0.836963$
$f(z_3,z_4,z_5) = -4.418$

3rd diff:
$f(z_0,z_1,z_2,z_3) = -0.297$
$f(z_1,z_2,z_3,z_4) = -0.39287$
$f(z_2,z_3,z_4,z_5) = -52.48763$

$f(z_0,z_1,z_2,z_3,z_4) = -0.47935$
$f(z_1,z_2,z_3,z_4,z_5) = -260.4738$

$f(z_0,z_1,z_2,z_3,z_4,z_5) = -1299.97225$

$$f(z_1,z_2) = \frac{-0.56079734 + 0.29004996}{0.2 - 0.1}$$

$$f(z_3,z_4) = \frac{-0.81401972 + 0.56079734}{0.3 - 0.2}$$

$$f(z_0,z_1,z_2) = \frac{-2.7074738 + 2.8019975}{0.2 - 0.1}$$

$$f(z_1,z_2,z_3) = \frac{-2.6159201 + 2.7074738}{0.2 - 0.1}$$

$$f(z_2,z_3,z_4) = \frac{-2.5322238 + 2.6159201}{0.3 - 0.2}$$

$$f(z_3,z_4,z_5) = \frac{-2.9734038 + 2.5322238}{0.3 - 0.2}$$

$$f(z_0,z_1,z_2,z_3) = \frac{0.915537 - 0.945237}{0.2 - 0.1}$$

$$f(z_1,z_2,z_3,z_4) = \frac{0.836963 - 0.915537}{0.3 - 0.1}$$

$$f(z_2,z_3,z_4,z_5) = \frac{-4.418 - 0.836963}{0.3 - 0.2}$$

$x_2 = z_4 = 0.3$ $\quad f(z_4) = -0.81401972$
$x_2 = z_5 = 0.3$ $\quad f(z_5) = -0.81401972$

4th diff

$$f(z_0,z_1,z_2,z_3,z_4) = \frac{-0.39287 + 0.297}{0.3 - 0.1}$$

$$f(z_1,z_2,z_3,z_4,z_5) = \frac{-52.48763 + 0.39287}{0.3 - 0.1}$$

$f(z_0,z_1,z_2,z_3,z_4) = -0.47935$
$f(z_1,z_2,z_3,z_4,z_5) = -260.4738$

5th diff

$$f(z_0,z_1,z_2,z_3,z_4,z_5) = \frac{-260.4738 + 0.47935}{0.3 - 0.1} = -1299.97225$$

$= -260.4738$

poly

$$H_5(x) = f(z_0) + f(z_0,z_1)(x-z_0) + f(z_0,z_1,z_2)(x-z_0)(x-z_1) + f(z_0,z_1,z_2,z_3)(x-z_0)(x-z_1)(x-z_2) + f(z_0,z_1,z_2,z_3,z_4)(x-z_0)(x-z_1)(x-z_2)(x-z_3) + f(z_0,z_1,z_2,z_3,z_4,z_5)(x-z_0)(x-z_1)(x-z_2)(x-z_3)(x-z_4)$$

=

Ex 3.4 Q2C

$$H_5(x) = -0.29004996 + (-2.8019975)(x-0.1) + 0.945237(x-0.1)(x-0.1) + (-0.297)(x-0.1)(x-0.1)(x-0.2) + (-0.47935)(x-0.1)^2(x-0.2)^2 - 1299.97225(x-0.1)^2(x-0.2)^2(x-0.3)$$

3. The data in Exercise 1 were generated using the following functions. Use the polynomials constructed in Exercise 1 for the given value of x to approximate $f(x)$ and calculate the absolute error.

a. $f(x) = x \ln x$; approximate $f(8.4)$.

b. $f(x) = \sin(e^x - 2)$; approximate $f(0.9)$.

c. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$; approximate $f(-1/3)$.

d. $f(x) = x \cos x - 2x^2 + 3x - 1$; approximate $f(0.25)$.

3) a)

3a $H_3(x) = 17.56492 + 3.116256(x-8.3) + 0.05948(x-8.3)^2 - 0.002022(x-8.3)^2(x-8.6)$

from Q1a

$f(x) = x \ln x$, $f(8.4)$

Approx value

$H_3(8.4) = 17.56492 + 3.116256(8.4-8.3) + 0.05948(8.4-8.3)^2 - 0.002022(8.4-8.3)^2(8.4-8.6)$

$= 17.8771323$

Exact value $f(8.4) = 8.4 \ln(8.4) = 17.87714$

$A.E = |f(8.4) - H_3(8.4)| = |17.87714 - 17.8771323|$

$= 0.00000769$