3.3 Divided Differences

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Suppose that $P_n(x)$ is the nth interpolating polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \ldots, x_n$. Although this polynomial is unique, there are alternate algebraic representations that are useful in certain situations. The divided differences of f with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}), \quad (3.5)$$

for appropriate constants $a_0, a_1, \ldots, a_n$. To determine the first of these constants, a_0, note that if $P_n(x)$ is written in the form of Eq. (3.5), then evaluating $P_n(x)$ at x_0 leaves only the constant term a_0; that is,

$$a_0 = P_n(x_0) = f(x_0).$$

Similarly, when $P(x)$ is evaluated at x_1, the only nonzero terms in the evaluation of $P_n(x_1)$ are the constant and linear terms,

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1);$$

so,

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (3.6)$$

Table 3.9 Any degree asked, u need to complete the table

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
		$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \dfrac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
		$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			

Hence, the interpolating polynomial in Eq. (3.5) is

$$P_n(x) = \underbrace{f[x_0] + f[x_0, x_1](x - x_0)}_{\text{Deg 1}} + \underbrace{\cancel{a_2}^{\,f(x_0,x_1,x_2)}(x - x_0)(x - x_1)}_{\text{Degree 2}} + f(x_0, x_1, x_2, x_3)(x - x_0)(x - x_1)(x - x_2) + \cdots$$

Deg 3 (max)

$$+ \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

forward / backward

The difference between x are the same (h)

$h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 \Rightarrow h = x_{i+1} - x_i$

$s = \frac{x - x_0}{h}$ (mostly pos)

$s = \frac{x - x_n}{h}$ (backward)

Newton's Forward Differences [highlight top parts]

$$P_n(x) = P_n(x_0 + sh) = \underbrace{f[x_0] + sh\,f[x_0, x_1]}_{\text{Degree 1}} + s(s-1)h^2 f[x_0, x_1, x_2] + s(s-1)(s-2)h^3 f(x_0, x_1, x_2, x_3)$$

Deg 2

Deg 3

$$+ \cdots + s(s-1)\cdots(s-n+1)h^n f[x_0, x_1, \ldots, x_n]$$

Newton's Backward Differences [highlight the bottom part]

$$P_n(x) = P_n(x_n + sh)$$

last value

$$= f[x_n] + shf[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}] + \cdots$$
$$+ s(s+1)\cdots(s+n-1)h^n f[x_n, \ldots, x_0].$$

$s = \frac{X - X_n}{h}$ (mostly neg)

EXERCISE SET 3.3

Solve Q1 & 2 using Divided Difference Rule

1. Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   - a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
   - b. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$
2. Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   - a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
   - b. $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$
3. Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   - a. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

     difference $= 0.25 = h$ $\quad$ $h = 0.25$ $\quad$ $h = 0.25$
   - b. $f(0.25)$ if $f(0.1) = -0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$

     $h = 0.1$ $\quad$ $h = 0.25$ $\quad$ $h = 0.25$

3. Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

a. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

$x = -\frac{1}{3}$, x_0, $f(x_0)$, x_1, x_2, x_3

* The highlighted part are the ones that will be used for highlighting the polynomial

$x_0 = -0.75$ $\quad f(x_0) = -0.0718125$

$f(x_0, x_1) = 0.18825$

$x_1 = -0.5$ $\quad f(x_1) = -0.02475$

$f(x_0, x_1, x_2) = 2.501$

$f(x_1, x_2) = 1.43875$

$f(x_0, x_1, x_2, x_3) = 1$

$x_2 = -0.25$ $\quad f(x_2) = 0.3349375$

$f(x_1, x_2, x_3) = 3.251$

$f(x_2, x_3) = 3.06425$

$x_3 = 0$ $\quad f(x_3) = 1.101$

$$s = \frac{x - x_0}{h} = \frac{-\frac{1}{3} - (-0.75)}{0.25} \qquad h = -0.5 + 0.75 = 0.25$$

$$-0.25 + 0.5 = 0.25$$

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{h} = \frac{-0.02475 + 0.0718125}{0.25} = 0.18825$$

$$f(x_1, x_2) = \frac{0.3349375 + 0.02425}{0.25} = 1.43875$$

$$f(x_2, x_3) = \frac{1.101 - 0.3349375}{0.25} = 3.06425$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{2h} = \frac{1.43875 - 0.18825}{2(0.25)} = 2.501$$

$$f(x_1, x_2, x_3) = \frac{3.06425 - 1.43875}{2(0.25)} = 3.251$$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{3h} = 1$$

$$s = \frac{x - x_0}{h} = \frac{-\frac{1}{3} + 0.75}{0.25} = \frac{5}{3}$$

when writing the polynomial, keep s in the formula but change h

Polynomial of deg 1:

$$P_1(x) = f(x_0) + sh\, f(x_0, x_1)$$
$$= -0.0718125 + s(0.25)(0.11825)$$

Approx value:

$$P_1\left(-\tfrac{1}{3}\right) = -0.0718125 + \left(\tfrac{5}{3}\right)(0.25)(0.11825) = 0.006625$$

3. Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

a. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

Poly deg 2:

$$P_2(x) = f(x_0) + sh\, f(x_0, x_1) + s(s-1)h^2 f(x_0, x_1, x_2)$$

$$= -0.071825 + s(0.25)(0.18825) + s(s-1)(0.25)^2(2.501)$$

Approx value: $P_2(-\frac{1}{3}) = -0.071825 + s(0.25)(0.18825) + \frac{5}{3}(\frac{5}{3}-1)(0.25)^2(2.501) = 0.180306$

Deg 3:

$$P_3(x) = f(x_0) + sh\, f(x_0, x_1) + s(s-1)h^2 f(x_0, x_1, x_2) + s(s-1)(s-2)h^3 f(x_0, x_1, x_2, x_3)$$

$$= -0.071825 + s(0.25)(0.18825) + \frac{5}{3}(\frac{5}{3}-1)(0.25)^2(2.501) + \frac{5}{3}(\frac{5}{3}-1)(\frac{5}{3}-2)(0.25)^3(1) = 0.174519$$

5. Use the Newton backward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

a. $f(-1/3)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$ (similar question to 3 however its backward) [same table and values but different polynomial formula and highlighted values are different]

$x_0 = -0.75$	$f(x_0) = -0.0718125$			
		$f(x_0, x_1) = 0.18825$		
$x_1 = -0.5$	$f(x_1) = -0.02475$		$f(x_0, x_1, x_2) = 2.501$	
		$f(x_1, x_2) = 1.43875$		$f(x_0, x_1, x_2, x_3) = 1$
$x_2 = -0.25$	$f(x_2) = 0.3349375$		$f(x_1, x_2, x_3) = 3.251$	
		$f(x_2, x_3) = 3.06425$		
$x_3 = 0$	$f(x_3) = 1.101$			

$$S = \frac{x - x_n}{h} = \frac{-\frac{1}{3} - 0}{0.25} = -\frac{4}{3}$$ (s is neg unlike forward)

Deg 1:

$P_1(x) = f(x_3) + sh f(x_3, x_2) = 1.101 + s(0.25)(3.06425)$

Approx value: $P_1(-\frac{1}{3}) = 1.01 + (-\frac{4}{3})(0.25)(3.06425) = 0.079583$

Deg 2:

$P_1(x) = f(x_3) + sh f(x_3, x_2) + s(s+1) h^2 f(x_3, x_2, x_1) = 1.101 + s(0.25)(3.06425) + s(s+1)(0.25)^2 (3.251)$

Approx value: $P_1(-\frac{1}{3}) = 1.01 + (-\frac{4}{3})(0.25)(3.06425) + (-\frac{4}{3})(-\frac{4}{3}+1)(0.25)^2(3.251) = 0.169889$

Deg 3:

$P_1(x) = f(x_3) + sh f(x_3, x_2) + s(s+1) h^2 f(x_3, x_2, x_1) + s(s+1)(s+2) h^3 f(x_0, x_1, x_2, x_3) = 1.101 + s(0.25)(3.06425) + s(s+1)(0.25)^2(3.251) + s(s+1)(s+2)(0.25)^3(1)$

Approx value: $P_1(-\frac{1}{3}) = 1.01 + (-\frac{4}{3})(0.25)(3.06425) + (-\frac{4}{3})(-\frac{4}{3}+1)(0.25)^2(3.251) + (-\frac{4}{3})(-\frac{4}{3}+1)(-\frac{4}{3}+2)(0.25)^3(1) = 0.174518$