3.2
Data Approximation and Neville's Method

Neville's Method

Disadvantage: No polynomial, we'll find the value only (unlike lagranche)

Theorem 3.5 Let f be defined at $x_0, x_1, \ldots, x_k$ and let x_j and x_i be two distinct numbers in this set. Then

$$P(x) = \frac{(x - x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x - x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i - x_j)}$$

is the kth Lagrange polynomial that interpolates f at the $k + 1$ points $x_0, x_1, \ldots, x_k$.

$$P_{0,1} = \frac{1}{x_1 - x_0}[(x - x_0)P_1 + (x - x_1)P_0], \quad P_{1,2} = \frac{1}{x_2 - x_1}[(x - x_1)P_2 + (x - x_2)P_1],$$

$$P_{0,1,2} = \frac{1}{x_2 - x_0}[(x - x_0)P_{1,2} + (x - x_2)P_{0,1}],$$

		Deg 1:	Deg 2:	Deg 3:	Deg 4:
x_0	P_0 $f(x_0)$				
x_1	P_1 $f(x_1)$	$P_{0,1}$			
x_2	P_2 $f(x_2)$	$P_{1,2}$	$P_{0,1,2}$		
x_3	P_3 $f(x_3)$	$P_{2,3}$	$P_{1,2,3}$	$P_{0,1,2,3}$	
x_4	P_4 $f(x_4)$	$P_{3,4}$	$P_{2,3,4}$	$P_{1,2,3,4}$	$P_{0,1,2,3,4}$

$$P_{01}(x) = \frac{(x-x_0)P_1 - (x-x_1)P_0}{x_1 - x_0}$$

$$P_{12}(x) = \frac{(x-x_1)P_2 - (x-x_2)P_1}{x_2 - x_1}$$

Degree 1: Degree 2: Deg 3:

		Degree 1:	Degree 2:	Deg 3:
x_0	$f(x_0) = P_0$			
		P_{01}		
x_1	$f(x_1) = P_1$		P_{012}	
		P_{12}		P_{0123}
x_2	$f(x_2) = P_2$		P_{123}	
		P_{23}		
x_3	$f(x_3) = P_3$			

$$P_{012}(x) = \frac{(x-x_0)P_{12} - (x-x_2)P_{01}}{x_2 - x_0}$$

drop the middle value

$$P_{123}(x) = \frac{(x-x_1)P_{23} - (x-x_3)P_{12}}{x_3 - x_1}$$

$$P_{0123}(x) = \frac{(x-x_0)P_{123} - (x-x_3)P_{012}}{x_3 - x_0}$$

	D1	D2	D3	D4
x_0 $\quad P_0 = f(x_0)$				
	P_{01}			
x_1 $\quad P_1 = f(x_1)$		P_{012}		
	P_{12}		P_{0123}	
x_2 $\quad P_2 = f(x_2)$		P_{123}		P_{01234}
	P_{23}		P_{1234}	
x_3 $\quad P_3 = f(x_3)$		P_{234}		
	P_{34}			
x_4 $\quad P_4 = f(x_4)$				

(max to be in an exam)

Deg 1

$$P_{01}(x) = \frac{(x-x_0)P_1 - (x-x_1)P_0}{x_1 - x_0}$$

$$P_{12}(x) = \frac{(x-x_1)P_2 - (x-x_2)P_1}{x_2 - x_1}$$

$$P_{23}(x) = \frac{(x-x_2)P_3 - (x-x_3)P_2}{x_3 - x_2}$$

$$P_{34}(x) = \frac{(x-x_3)P_4 - (x-x_4)P_3}{x_4 - x_3}$$

Deg 2

$$P_{012}(X) = \frac{(X-X_0)P_{12} - (X-X_2)P_{01}}{X_2 - X_0}$$

$$P_{123}(X) = \frac{(X-X_1)P_{23} - (X-X_3)P_{12}}{X_3 - X_1}$$

$$P_{234}(X) = \frac{(X-X_2)P_{34} - (X-X_4)P_{23}}{X_4 - X_2}$$

Degree 3:

$$P_{0123}(x) = \frac{(x-x_0)P_{123} - (x-x_3)P_{012}}{x_3 - x_0}$$

$$P_{1234}(x) = \frac{(x-x_0)P_{234} - (x-x_4)P_{123}}{x_4 - x_0}$$

Degree 4:

$$P_{01234}(x) = \frac{(x-x_0)P_{1234} - (x-x_4)P_{0123}}{x_4 - x_0}$$

x_0	$P_0 = Q_{0,0}$				
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$		
x_3	$P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$	
x_4	$P_4 = Q_{4,0}$	$P_{3,4} = Q_{4,1}$	$P_{2,3,4} = Q_{4,2}$	$P_{1,2,3,4} = Q_{4,3}$	$P_{0,1,2,3,4} = Q_{4,4}$

EXERCISE SET 3.2

1. Use Neville's method to obtain the approximations for Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:
   a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
   b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
   c. $f(0.25)$ if $f(0.1) = 0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$
   d. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$

1. Use Neville's method to obtain the approximations for Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

x_0 P_0 x_1 P_1 x_2 P_2 x_3 P_3

$x_0 = -0.75$ $P_0 = -0.0718125$

$x_1 = -0.5$ $P_1 = -0.02475$ $P_{01} = 0.06625$ $P_{012} = 0.1863055556$

$-\frac{1}{3}$ (between)

$x_2 = -0.25$ $P_2 = 0.3349375$ $\boxed{P_{12} = 0.2150416667}$ choose the value in between

choose either

$\boxed{P_{123} = 0.1698889}$ $\boxed{P_{0123} = 0.1745185\text{2}}$

$x_3 = 0$ $P_3 = 1.101$ $P_{23} = 0.0795833333$

$$P_{01}\left(-\frac{1}{3}\right) = \frac{\left(-\frac{1}{3}+0.75\right)(-0.02475) - \left(-\frac{1}{3}+0.5\right)(-0.0718125)}{-0.5-(-0.75)} = 0.06625$$

$$P_{12}\left(-\frac{1}{3}\right) = \frac{\left(-\frac{1}{3}+0.5\right)(0.3349375) - \left(-\frac{1}{3}+0.25\right)(-0.02475)}{-0.25+0.5} = 0.2150416667$$

$$P_{23}\left(-\frac{1}{3}\right) = \frac{\left(-\frac{1}{3}+0.25\right)(1.101) - \left(-\frac{1}{3}-0\right)(0.3349375)}{0+0.25} = 0.0795833333$$

Deg 2°

$$P_{012} = \frac{\left(-\frac{1}{3}+0.75\right)(0.2150416667)-\left(-\frac{1}{3}+0.25\right)(0.606625)}{-0.25+0.75} = 0.1803055556$$

$$P_{123} = \frac{\left(-\frac{1}{3}+0.5\right)(0.07958333333)-\left(-\frac{1}{3}-0\right)(0.2150416667)}{0+0.5} = 0.1698889$$

Deg 3°

$$P_{0123} = \frac{\left(-\frac{1}{3}+0.75\right)(0.1698889)-\left(-\frac{1}{3}-0\right)(0.1803055556)}{0+0.75} = 0.17451852$$

3. Use Neville's method to approximate $\sqrt{3}$ with the following functions and values.

a. $f(x) = 3^x$ and the values $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 2$.

b. $f(x) = \sqrt{x}$ and the values $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 5$.

c. Compare the accuracy of the approximation in parts (a) and (b).

4. Let $P_3(x)$ be the interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$, and $(2, 2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 6.

5. Neville's method is used to approximate $f(0.4)$, giving the following table.

$x_0 = 0$	$P_0 = 1$			
$x_1 = 0.25$	$P_1 = 2$	$P_{01} = 2.6$		
$x_2 = 0.5$	P_2	$P_{1,2}$	$P_{0,1,2}$	
$x_3 = 0.75$	$P_3 = 8$	$P_{2,3} = 2.4$	$P_{1,2,3} = 2.96$	$P_{0,1,2,3} = 3.016$

Determine $P_2 = f(0.5)$.

6. Neville's method is used to approximate $f(0.5)$, giving the following table.

$x_0 = 0$	$P_0 = 0$		
$x_1 = 0.4$	$P_1 = 2.8$	$P_{0,1} = 3.5$	
$x_2 = 0.7$	P_2	$P_{1,2}$	$P_{0,1,2} = \frac{27}{7}$

Determine $P_2 = f(0.7)$.

7. Suppose $x_j = j$, for $j = 0, 1, 2, 3$, and it is known that

$$P_{0,1}(x) = 2x + 1, \quad P_{0,2}(x) = x + 1, \quad \text{and} \quad P_{1,2,3}(2.5) = 3.$$

1,2

Find $P_{0,1,2,3}(2.5)$.

$\sqrt{3} = 3^x \Rightarrow 3^{\frac{1}{2}} = 3^x \Rightarrow x = \frac{1}{2}$

3. Use Neville's method to approximate $\sqrt{3}$ with the following functions and values.

a. $f(x) = 3^x$ and the values $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 2$.

$$
\begin{array}{llll}
x_0 = -2 & P_0 = 3^{-2} = \frac{1}{9} & & \\
 & & P_{01} = \frac{2}{3} & \\
x_1 = -1 & P_1 = 3^{-1} = \frac{1}{3} & & P_{012} = \frac{3}{2} \\
 & & P_{12} = \frac{4}{3} & \\
x_2 = 0 & P_2 = 3^0 = 1 & & \\
 & & P_{23} & \\
x_3 = 1 & P_3 = 3^1 = 3 & & \\
 & & P_{34} & \\
x_4 = 2 & P_4 = 3^2 = 9 & &
\end{array}
$$

$(x - x_0)\,P$

$$P_{01} = \frac{(\frac{1}{2}+2)\frac{1}{3} - (\frac{1}{2}+1)\frac{1}{9}}{-1+2} = \frac{2}{3}$$

$$P_{012} = \frac{(\frac{1}{2}+2)\frac{4}{3} - (\frac{1}{2}-0)\frac{2}{3}}{0+2} = \frac{3}{2}$$

$$P_{12} = \frac{(\frac{1}{2}+1)(1) - (\frac{1}{2}-0)\frac{1}{3}}{0+1} = \frac{4}{3}$$

b. $f(x) = \sqrt{x}$ and the values $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 5$.

$f(x) = \sqrt{x} = \sqrt{3}$

$x = 3$

$x_0 = 0$ $\quad P_0 = \sqrt{0} = 0$

$x_1 = 1$ $\quad P_1 = \sqrt{1} = 1$

$P_{01} = 3$

$x_2 = 2$ $\quad P_2 = \sqrt{2}$

$x_3 = 4$ $\quad P_3 = \sqrt{4} = 2$

$x_4 = 5$ $\quad P_4 = \sqrt{5}$

$$P_{01} = \frac{(3-0)1 - (3-1)0}{1-0} = 3$$

5. Neville's method is used to approximate $f(0.4)$, giving the following table.

$x_0 = 0$	$P_0 = 1$			
$x_1 = 0.25$	$P_1 = 2$	$P_{01} = 2.6$		
$x_2 = 0.5$	$P_2 = 4$	$P_{1,2} = 3.2$	$P_{0,1,2} = 3.08$	
$x_3 = 0.75$	$P_3 = 8$	$P_{2,3} = 2.4$	$P_{1,2,3} = 2.96$	$P_{0,1,2,3} = 3.016$

$f(x_2)$ Start backwords to find P_2

Determine $P_2 = f(0.5)$.

$$P_{0123} = \frac{(x - x_0) P_{123} - (x - x_3) P_{012}}{x_3 - x_0}$$

$$3.016 = \frac{(0.4 - 0)\,2.96 - (0.4 - 0.75)\, P_{012}}{0.75 - 0}$$

$P_{012} = 3.08$ using calculator

$$P_{012} = \frac{(x - x_0) P_{12} - (x - x_2) P_{01}}{x_2 - x_0}$$

$$3.08 = \frac{(0.4 - 0) P_{12} - (0.4 - 0.5)(2.6)}{0.5 - 0}$$

$P_{12} = 3.2$

$$P_{12} = \frac{(x - x_1) P_2 - (x - x_2) P_1}{x_2 - x_1}$$

$$3.2 = \frac{(0.4 - 0.25) P_2 - (0.4 - 0.5)(2)}{0.5 - 0.25}$$

$P_2 = 4$

7. Suppose $x_j = j$, for $j = 0, 1, 2, 3$, and it is known that

$$P_{0,1}(x) = 2x + 1, \qquad P_{0,2}(x) = x + 1, \qquad \text{and} \qquad P_{1,2,3}(2.5) = 3.$$

Find $P_{0,1,2,3}(2.5)$.

2.5   1,2   2.5+1   3.5

$P_{0,1}(x) = 2(2.5)+1 = 6$

$x = 2.5$

$$P_{01} = \frac{(x-x_0)P_1-(x-x_1)P_0}{x_1-x_0}$$

$$P_{0,1,2} = \frac{(x-x_0)P_{12}-(x-x_2)P_{01}}{x_2-x_0} = \frac{(2.5-0)3.5-(2.5-2)(6)}{2-0} =$$

$x_0 = 0$ P_0

$x_1 = 1$ P_1

$x_2 = 2$ P_2

$x_3 = 3$ P_3

$P_{01}(2.5) = 2(2.5)+1 = 6$

$P_{12}(2.5) = 2.5+1 = 3.5$

$P_{23}(2.5) =$

$P_{012} =$

$P_{123} = 3$

$P_{0123} = ?$

$$P_{012} = \frac{(x-x_0)P_{12}-(x-x_2)P_{01}}{x_2-x_0}$$

$$P_{012} = \frac{(2.5-0)(3.5)-(2.5-2)}{2-0}$$

$$P_{0123} = \frac{(x-x_0)P_{123}-(x-x_3)P_{012}}{x_3-x_0}$$