Ch : 3 → create functions

CH-3.1

Interpolation

Pol

find the type of function depending on the graph with different points

Lagrange polynomial:

The problem of determining a polynomial of degree one that passes through the distinct points (x_0, y_0) and (x_1, y_1) is the same as approximating a function f for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first-degree polynomial **interpolating**, or agreeing with, the values of f at the given points. Using this polynomial for approximation within the interval given by the endpoints is called polynomial **interpolation**.

2 points

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

* only when 2 points are given

$P_1(x) = f(x_0)L_0(x) + f(x_1)L_1(x)$

Pol

The linear **Lagrange interpolating polynomial** through (x_0, y_0) and (x_1, y_1) is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Note that

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_1(x_0) = 0, \quad \text{and} \quad L_1(x_1) = 1,$$

which implies that

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

and

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1.$$

So P is the unique polynomial of degree at most one that passes through $(x_0,$ (x_1, y_1).

Exa

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Solution In this case we have

$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \quad \text{and} \quad L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2),$$

so

$$P(x) = -\frac{1}{3}(x-5) \cdot 4 + \frac{1}{3}(x-2) \cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x + 6.$$

$x_0 \quad x_1 \quad x_2$

Degree 1

$$L_0 = \frac{x - x_1}{x_0 - x_1}, \quad L_1 = \frac{x - x_0}{x_1 - x_0}$$

$$p(x) = \underline{L_0(x)}\ \underline{f(x_0)} + \underline{L_1(x)}\ \underline{f(x_1)}$$

Degree 2

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \quad L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$p(x) = L_0(x)\, f(x_0) + L_1 f(x_1) + L_2 f(x_2)$$

Degree 3 (max deg asked in the exam)

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}, \quad L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}, \quad L_2(x) = \frac{(x-x)(x-x)(x-x)}{(x-x)(x-x)(x-x)}, \quad L_3(x) = \frac{(x-x)(x-x)(x-x)}{(x-x)(x-x)(x-x)}$$

$$\underline{L_0(x)} \times \underline{f(x_0)} \qquad f(x) = \frac{1}{x} \qquad x_0 = 2 \quad x_1 = 2.75 \quad x_2 = 4$$

If more than 2 points are given, to create degree 1 polynomial choose 2 points, however if degree 2 is needed choose 3 points

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k), \quad \text{for each } k = 0, 1, \ldots, n.$$

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x), \tag{3.1}$$

where, for each $k = 0, 1, \ldots, n$,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \tag{3.2}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}.$$

We will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree.

Exa

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{(x-2.75)(x-4)}{1.5}$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = \frac{(x-2)(x-4)}{-0.9375}$$

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{(x-2)(x-2.75)}{2.5}$$

$$P(x) = \frac{1}{2}\left[\frac{(\ -2.75)(\ -4)}{1.5}\right] + \frac{1}{2.75}\left[\frac{(\ -2)(\ -4)}{-0.9375}\right] + \frac{1}{4}\left[\frac{(x-2)(x-2.75)}{2.5}\right] = \underbrace{0.32955}$$

about $\frac{1}{3} = 0.3333$

(a) Use the numbers (called *nodes*) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$. $f(x_0) = \frac{1}{2}$ $f(x_1) = \frac{1}{2.75}$ $f(x_2) = \frac{1}{4}$

(b) Use this polynomial to approximate $f(3) = 1/3$.

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_1)} = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)}$$

Solution **(a)** We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$. In nested form they are

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.5)(2-4)} = \frac{2}{3}(x-2.75)(x-4),$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4),$$

and

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.5)} = \frac{2}{5}(x-2)(x-2.75).$$

Also, $f(x_0) = f(2) = 1/2$, $f(x_1) = f(2.75) = 4/11$, and $f(x_2) = f(4) = 1/4$, so

$$\begin{aligned} P(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\ &= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\ &= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}. \end{aligned}$$

(b) An approximation to $f(3) = 1/3$ (see Figure 3.6) is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

Recall that in the opening section of this chapter (see Table 3.1) we found that no Taylor polynomial expanded about $x_0 = 1$ could be used to reasonably approximate $f(x) = 1/x$ at $x = 3$. ■

Exercise Set 3.1

1. For the given functions $f(x)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$, and find the absolute error.

a. $f(x) = \cos x$

b. $f(x) = \sqrt{1+x}$

c. $f(x) = \ln(x+1)$

d. $f(x) = \tan x$

$L_0 = \frac{x-x_1}{x_0-x_1} = \frac{0.45-0.6}{0-0.6} = \frac{1}{4}$

$L_1 = \frac{x-x_0}{x_1-x_0} = \frac{0.45-0}{0.6-0} = \frac{3}{4}$

$\frac{1}{4} \times \ln(0+1) + \frac{3}{4} \times \ln(0.6+1)$

$= 0.352503$

$L_0 = \frac{(x-x_1)}{(x_0-x_1)} = \frac{(0.45-0.6)}{0-0.6} = \frac{1}{4}$

$L_1 = \frac{(x-x_0)}{(x_1-x_0)} = \frac{(0.45-0)}{0.6-0} = \frac{3}{4}$

$p(x) =$

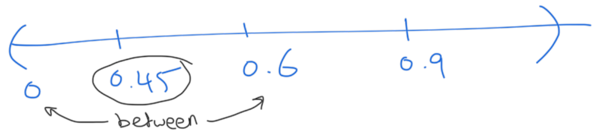
1. For the given functions $f(x)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$, and find the absolute error.

a. $f(x) = \cos x$ **c.** $f(x) = \ln(x + 1)$

Deg 1: $x_0 = 0,\ x_1 = 0.6$

$f(x_0) = \cos 0 = 1$

$f(x_1) = \cos(0.6)$



So take 0 & 0.6 for degree 1

$$L_0(x) = \frac{x-0.6}{0-0.6} = \frac{x-0.6}{-0.6}$$

$$L_1(x) = \frac{x-0}{0.6-0} = \frac{x}{0.6}$$

$$P(x) = \frac{x-0.6}{-0.6}(1) + \frac{x}{0.6}\cos(0.6)$$

$$P(0.45) = \frac{0.45-0.6}{-0.6} + \frac{0.45}{0.6}\cos(0.6) = 0.869002$$

Exact value: $\cos(0.45) = 0.90045$

Absolute error $= |0.90045 - 0.869002| = 0.031448$

The more the degree is, the difference in absolute error is less

Deg 2:

$$L_0(x) = \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)}$$

$$L_1(x) = \frac{(x-0)(x-0.9)}{(0.6-0)(0.6-0.9)}$$

$$L_2(x) = \frac{(x-0)(x-0.6)}{(0.9-0)(0.9-0.6)}$$

$$P(x) = \frac{(x-0.6)(x-0.9)}{0.54}(1) + \frac{(x(x-0.9))}{-0.18}(\cos(0.6)) + \frac{x(x-0.6)}{0.27}\cos(0.9)$$

$$P(x) = \frac{(0.45-0.6)(0.45-0.9)}{0.54}(1) + \frac{(0.45(0.45-0.9))}{-0.18}(\cos(0.6)) + \frac{0.45(0.45-0.6)}{0.27}\cos(0.9) = 0.8981$$

Exact value: $\cos(0.45) = 0.90045$

Absolute error $= |0.90045 - 0.8981| = 0.0023$

Exercise

create/find the polynomial first then find the value

5. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

c. $f(0.25)$ if $f(0.1) = 0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$

d. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$

5. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

$-0.25 < f\left(-\frac{1}{3}\right) < -0.5$

Deg 1: $x_0 = -0.5 \quad x_1 = -0.25$

$f(-0.5) = -0.02475$

$f(-0.25) = 0.3349375$

$L_0(x) = \frac{(x+0.25)}{-0.5-(-0.25)} = \frac{(x+0.25)}{-0.25}$

$L_1(x) = \frac{(x+0.5)}{-0.25+0.5} = \frac{(x+0.5)}{0.25}$

$p(x) = L_0(x) \cdot f(x_0) + L_1(x) \times f(x_1) = \frac{x+0.25}{-0.25}(-0.02475) + \frac{x+0.5}{0.25}(0.33975)$

$p\left(-\frac{1}{3}\right) = L_0(x) \cdot f(x_0) + L_1(x) \times f(x_1) = \frac{-\frac{1}{3}+0.25}{-0.25}(-0.02475) + \frac{-\frac{1}{3}+0.5}{0.25}(0.33975) = 0.2$

Deg 2: $x_0 = -0.75 \quad x_1 = -0.5 \quad x_2 = -0.25$ (u can choose the first 3 values or the next 3 values, both would be correct)

☆ If u want, choose the points that has a zero as it'll be easier

$f(-0.75) = -0.0718125$

$f(-0.5) = -0.02475$

$f(-0.25) = 0.3349375$

$L_0(x) = \frac{(x+0.5)(x+0.25)}{(-0.75+0.5)(-0.75+0.25)}$

$L_1(x) = \frac{(x+0.75)(x+0.25)}{(-0.5+0.75)(-0.5+0.25)}$

$L_2(x) = \frac{(x+0.75)(x+0.5)}{(-0.25+0.75)(-0.25+0.5)}$

$p(x) = L_0(x) \cdot f(x_0) + L_1(x) \times f(x_1) + L_2(x) \times f(x_2)$

$p\left(-\frac{1}{3}\right) = \frac{(-\frac{1}{3}+0.5)(-\frac{1}{3}+0.25)}{(-0.75+0.5)(-0.75+0.25)}(-0.0718125) + \frac{(-\frac{1}{3}+0.75)(-\frac{1}{3}+0.25)}{(-0.5+0.75)(-0.5+0.25)}(-0.02475) + \frac{(-\frac{1}{3}+0.75)(-\frac{1}{3}+0.5)}{(-0.25+0.75)(-0.25+0.5)}(0.33975)$

$= 0.00797917 - 0.01375 + 0.1860763889$

$= 0.180303056$

5. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

Deg 3: $x_0 = -0.75,\ x_1 = -0.5,\ x_2 = -0.25,\ x_3 = 0$

$f(-0.75) = -0.0718125$
$f(-0.5) = -0.02475$
$f(-0.25) = 0.3349375$
$f(0) = 1.101$

$$L_0(x) = \frac{(x+0.5)(x+0.25)(x-0)}{(-0.75+0.5)(-0.75+0.25)(-0.75-0)}$$

$$L_1(x) = \frac{(x+0.75)(x+0.25)(x-0)}{(-0.5+0.75)(-0.5+0.25)(-0.5-0)}$$

$$L_2(x) = \frac{(x+0.75)(x+0.5)(x-0)}{(-0.25+0.75)(-0.25+0.5)(-0.25-0)}$$

$$L_3(x) = \frac{(x+0.75)(x+0.5)(x+0.25)}{(0+0.75)(0+0.5)(0+0.25)}$$

$$p\left(-\tfrac{1}{3}\right) = L_0(x)\times f(x_0) + L_1(x)\times f(x_1) + L_2(x)\times f(x_2) + L_3(x)\times f(x_3)$$

$$= \frac{\left(-\frac{1}{3}+0.5\right)\left(-\frac{1}{3}+0.25\right)\left(-\frac{1}{3}-0\right)}{(-0.75+0.5)(-0.75+0.25)(-0.75-0)}(-0.0718125) + \frac{\left(-\frac{1}{3}+0.75\right)\left(-\frac{1}{3}+0.25\right)\left(-\frac{1}{3}-0\right)}{(-0.5+0.75)(-0.5+0.25)(-0.5-0)}(-0.02475)$$

$$+ \frac{\left(-\frac{1}{3}+0.75\right)\left(-\frac{1}{3}+0.5\right)\left(-\frac{1}{3}-0\right)}{(-0.25+0.75)(-0.25+0.5)(-0.25-0)}(0.3349375) + \frac{\left(-\frac{1}{3}+0.75\right)\left(-\frac{1}{3}+0.5\right)\left(-\frac{1}{3}+0.25\right)}{(0+0.75)(0+0.5)(0+0.25)}(1.101)$$

$$= 0.17451852$$