CHAPTER 2.5

Accelerating Convergence

Aitken's Δ^2 method is based on the assumption that the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, defined by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n},$$

3 p values are needed to calculate $\hat{p}$

EXERCISE SET 2.5

1. The following sequences are linearly convergent. Generate the first five terms of the sequence $\{\hat{p}_n\}$ using Aitken's Δ^2 method.

   $x = \boxed{}$ so $g(x)$

   a. $p_0 = 0.5, \quad p_n = (2 - e^{p_{n-1}} + p_{n-1}^2)/3, \quad n \geq 1$

   $P_1 = \frac{2 - e^{P_0} + P_0^2}{3} = \frac{2 - e^{0.5} + (0.5)^2}{3} = 0.200426$

   b. $p_0 = 0.75, \quad p_n = (e^{p_{n-1}}/3)^{1/2}, \quad n \geq 1$

   c. $p_0 = 0.5, \quad p_n = 3^{-p_{n-1}}, \quad n \geq 1$

   d. $p_0 = 0.5, \quad p_n = \cos p_{n-1}, \quad n \geq 1$

2. Consider the function $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$. Use Newton's method with $p_0 = 0$ to approximate a zero of f. Generate terms until $|p_{n+1} - p_n| < 0.0002$. Construct the sequence $\{\hat{p}_n\}$. Is the convergence improved?

3. Let $g(x) = \cos(x - 1)$ and $p_0^{(0)} = 2$. Use Steffensen's method to find $p_0^{(1)}$.

4. Let $g(x) = 1 + (\sin x)^2$ and $p_0^{(0)} = 1$. Use Steffensen's method to find $p_0^{(1)}$ and $p_0^{(2)}$.

5. Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1$ and $p_2^{(0)} = 3$ to obtain $p_0^{(1)} = 0.75$. What is $p_1^{(0)}$?

6. Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1$ and $p_1^{(0)} = \sqrt{2}$ to obtain $p_0^{(1)} = 2.7802$. What is $p_2^{(0)}$? H.W

7. Use Steffensen's method to find, to an accuracy of 10^{-4}, the root of $x^3 - x - 1 = 0$ that lies in $[1, 2]$, and compare this to the results of Exercise 6 of Section 2.2.

1) a)

$$P_1 = \frac{2 - e^{P_0} + P_0^2}{3} = \frac{2 - e^{0.5} + (0.5)^2}{3} = 0.200426$$

$$P_2 = \frac{2 - e^{P_1} + (P_1)^2}{3} = 0.272749$$

$$P_3 = \frac{2 - e^{P_2} + (P_2)^2}{3} = 0.253607$$

$$P_4 = \frac{2 - e^{P_3} + (P_3)^2}{3} = 0.25855$$

$$P_5 = \frac{2 - e^{P_4} + (P_4)^2}{3} = 0.257265$$

$$P_6 = \frac{2 - e^{P_5} + (P_5)^2}{3} = 0.257598$$

$$P_7 = \frac{2 - e^{P_6} + (P_6)^2}{3} = 0.257512$$

$$\hat{P}_0 = P_0 - \frac{(P_1 - P_0)^2}{P_2 - 2P_1 + P_0}$$

$$= 0.5 - \frac{(0.200426 - 0.5)^2}{0.272749 - 2(0.200426) + 0.5}$$

$$= 0.258684$$

$$\hat{P}_1 = 0.200426 - \frac{(0.272749 - 0.200479)^2}{0.253607 - 2(0.272749) + 0.200479}$$

$$= 0.257613$$

$$\hat{P}_2 = P_2 - \frac{(P_3 - P_2)^2}{P_4 - 2P_3 + P_2} = 0.257536$$

$$\hat{P}_3 = P_3 - \frac{(P_4 - P_3)^2}{P_5 - 2P_4 + P_3} = 0.257530$$

$$\hat{P}_4 = P_4 - \frac{(P_5 - P_4)^2}{P_6 - 2P_5 + P_4} = 0.257497$$

$$\hat{P}_5 = P_5 - \frac{(P_6 - P_5)^2}{P_7 - 2P_6 + P_5} = 0.257429$$

↑
the question
needs 5 terms
So we need upto P_7

1) c) $P_0 = 0.5 \qquad P_n = 3^{-P_{n-1}} \qquad n \geq 1$

$P_1 = 3^{-0.5} = 0.57735$

$P_2 = 3^{-P_1} = 0.530315$

$P_3 = 3^{-P_2} = 0.558439$

$P_4 = 3^{-P_3} = 0.541448$

$P_5 = 3^{-P_4} = 0.55165$

$P_6 = 3^{-P_5} = 0.545502$

$P_7 = 3^{-P_6} = 0.549199$

$$\hat{P}_0 = 0.5 - \frac{(0.57735 - 0.5)^2}{0.530315 - 2(0.57735) + 0.5} = 0.5481$$

$$\hat{P}_1 = P_1 - \frac{(P_2 - P_1)^2}{P_3 - 2P_2 + P_1} = 0.547915$$

$$\hat{P}_2 = P_2 - \frac{(P_3 - P_2)^2}{P_4 - 2(P_3) + P_2} = 0.547847$$

$$\hat{P}_3 = P_3 - \frac{(P_4 - P_3)^2}{P_5 - 2P_4 + P_3} = 0.547822$$

$$\hat{P}_4 = P_4 - \frac{(P_5 - P_4)^2}{P_6 - 2P_5 + P_4} = 0.547813$$

$$\hat{P}_5 = P_5 - \frac{(P_6 - P_5)^2}{P_7 - 2P_6 + P_5} = 0.54781$$

3. Let $g(x) = \cos(x-1)$ and $p_0^{(0)} = 2$. Use Steffensen's method to find $p_0^{(1)}$.

$$p_1^{(0)} = g(p_0^0) = \cos(2-1) = 0.540302$$

$$g(p_n)$$

$$p_0^0$$

$$p_2^{(0)} = g(p_1^0) = \cos(0.540302-1) = 0.896187$$

$$p_0^{(1)} = p_0^{(0)} - \frac{(p_1^{(0)} - p_0^{0})^2}{p_2^{(0)} - 2p_1^0 + p_0^0}$$

$$p_0^1 = p_0^0 - \frac{(p_1^0 - p_0^0)^2}{p_2^0 - 2p_1^0 + p_0^0} = 0.826428$$

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n},$$

4. Let $g(x) = 1 + (\sin x)^2$ and $p_0^{(0)} = 1$. Use Steffensen's method to find $p_0^{(1)}$ and $p_0^{(2)}$.

$$p_1^0 = g(p_0^0) = 1 + (\sin 1)^2 = 1.708073$$

$$p_0^{(2)} = p_0^1 - \frac{(p_1^1 - p_0^1)^2}{p_2^1 - 2p_1^1 + p_0^1}$$

$$p_2^0 = g(p_1^0) = 1 + (\sin(1.708073))^2 = 1.981273$$

$$p_0^1 = \frac{(1.708073-1)^2}{1.981273 - 2(1.708073) + 1} = 2.152905$$

$$p_1^1 = g(p_0^1) = 1 + (\sin(2.152905))^2 = 1.697735$$

$$p_2^1 = g(p_1^1) = 1 + (\sin(1.697735)) = 1.983972$$

$$p_0^2 = 2.152905 - \frac{(1.697735 - 2.152905)^2}{1.983972 - 2(1.697735) + 2.152905} = 1.873464$$

3. Let $g(x) = \cos(x-1)$ and $p_0^{(0)} = 2$. Use Steffensen's method to find $p_0^{(1)}$.

$p_0^0 \quad p_1^0 \quad p_2^0$

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n},$$

$$p_1^0 = \cos(2-1) = 0.540302$$

$$p_2^0 = \cos(0.540302-1) = 0.896187$$

$$p_0^1 = p_0^0 - \frac{(p_1^0 - p_0^0)^2}{p_2^0 - 2p_1^0 + p_0^0}$$

$$= 2 - \frac{(0.540302-2)^2}{0.89-2(0.54)+2} = 0.826428$$

5. Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1$ and $p_2^{(0)} = 3$ to obtain $p_0^{(1)} = 0.75$. What is $p_1^{(0)}$?

$$P_1^0 = P_0^0 - \frac{(P_1^0 - P_0^0)^2}{P_2^0 - 2P_1^0 + P_0^0}$$

$$0.75 = 1 - \frac{(P_1^0 - 1)^2}{3 - 2P_1^0 + 1}$$

$$0.75 - 1 = -\frac{(P_1^0 - 1)^2}{4 - P_1^0}$$

$$+0.25 = +\frac{(P_1^0 - 1)^2}{4 - P_1^0}$$

$$0.25 = \frac{(P_1^0 - 1)^2}{4 - P_1^0}$$

$$\cancel{1} - 0.5P_1^0 = (P_1^0)^2 - 2P_1^0 + \cancel{1}$$

$$-0.5P_1^0 = (P_1^0)^2 - 2P_1^0$$

$$(P_1^0)^2 - 1.5P_1^0 = 0$$

$$P_1^0(P_1^0 - 1.5) = 0$$

$$P_1^0 = 0 \text{ or } P_1^0 = 1.5$$

7. Use Steffensen's method to find, to an accuracy of 10^{-4}, the root of $x^3 - x - 1 = 0$ that lies in $[1, 2]$, and compare this to the results of Exercise 6 of Section 2.2.

$$x^3 - x - 1 = 0$$

$$\sqrt[3]{x^3} = \sqrt[3]{x+1}$$

$$x = (x+1)^{\frac{1}{3}}$$

$$g'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}$$

$$g'(1) = \left|\frac{1}{3}(1+1)^{-\frac{2}{3}}\right| = 0.2 \leq 1$$

$$g'(2) = \left|\frac{1}{3}(2+1)^{-\frac{2}{3}}\right| = 0.16 \leq 1$$

$$p_0^0 = 1.5$$

$$p_1^0 = g(p_0^0) = (1.5+1)^{\frac{1}{3}} = 1.357208$$

$$p_2^0 = g(p_1^0) = (1.357208+1)^{\frac{1}{3}} = 1.330861$$

$$p_0^1 = 1.5 - \frac{(1.357208 - 1.5)^2}{1.330861 - 2(1.357208) + 1.5} = 1.3249$$

$$p_1^1 = g(p_0^1) = (1.3249+1)^{\frac{1}{3}} = 1.324733$$

$$p_2^1 = g(p_1^1) = (1.324733+1)^{\frac{1}{3}} = 1.324721$$

$$p_0^2 = 1.3249 - \frac{(1.324733 - 1.3249)^2}{1.324721 - 2(1.324733) + 1.3249} = 1.324978$$

$$|p_0^2 - p_0^1| = |1.324978 - 1.3249| = 0.00007 \leq 10^{-4}$$

So p_0^2 is the solution

Steffensen's Method

By applying a modification of Aitken's Δ^2 method to a linearly convergent sequence obtained from fixed-point iteration, we can accelerate the convergence to quadratic. This procedure is known as Steffensen's method and differs slightly from applying Aitken's Δ^2 method directly to the linearly convergent fixed-point iteration sequence. Aitken's Δ^2 method constructs the terms in order:

$$p_0, \quad p_1 = g(p_0), \quad p_2 = g(p_1), \quad \hat{p}_0 = \{\Delta^2\}(p_0),$$

$$p_3 = g(p_2), \quad \hat{p}_1 = \{\Delta^2\}(p_1), \ldots,$$

when u plug the g function in the subscript $p_1^{0} = g(p_0^{0})$

where $\{\Delta^2\}$ indicates that Eq. (2.15) is used. Steffensen's method constructs the same first four terms, p_0, p_1, p_2, and $\hat{p}_0$. However, at this step we assume that $\hat{p}_0$ is a better approximation to p than is p_2 and apply fixed-point iteration to $\hat{p}_0$ instead of p_2. Using this notation, the sequence is

$$p_0^{(0)}, \quad p_1^{(0)} = g(p_0^{(0)}), \quad p_2^{(0)} = g(p_1^{(0)}), \quad p_0^{(1)} = \{\Delta^2\}(p_0^{(0)}), \quad p_1^{(1)} = g(p_0^{(1)}), \ldots.$$

Steffensen's method

p_0^0

$p_1^0 = g(p_0^0) = p_1^0$ when substituted in g fun ⇒ Superscript is always 0// doesnt change

$p_2^0 = g(p_1^0) = p_2^0$ change the subscript.

iterations p_0^1 (formula of steffensen) ⇒ Subscript is always 0// doesnt change

$$p_0^1 = p_0^0 - \frac{(p_1^0 - p_0^0)^2}{p_2^0 - 2p_1^0 + p_0^0}$$

(similar formula as Aitken's)

$$g(p_0^1) = p_1^1$$

$$p_0^2 = p_0^1 - \frac{(p_1^1 - p_0^1)^2}{p_2^1 - 2p_1^1 + p_0^1}, \quad g(p_0^2) = g(p_1^2)$$

$$p_0^3 = p_0^2 - \frac{(p_1^2 - p_0^2)^2}{p_2^2 - 2p_1^2 + p_0^2}$$

Guidelines

Step 1

When given an equation, for example $x^3 + 4x^2 - 10 = 0$, we must first solve for x. In this case, after solving for x, we will get

$$x = \sqrt{\frac{10}{x+4}}$$

We will then modify x to g(x) after solving for x to highlight that this is our fixed point iteration.

$$g(x) = \sqrt{\frac{10}{x+4}}$$

k	p_0^k	p_1^k	p_2^k
0			
1			
2			
3			

k	p_0^k	p_1^k	p_2^k
0	1.5	1.348399725	1.367376372
1			
2			
3			

Step 6

Next, for p0 k=1, we have to use the Steffensen's formula.

$$\hat{P} = P_n - \frac{(P_{n+1} - P_n)^2}{(P_{n+2} - 2P_{n+1} + P_n)}$$

Substitute all n value with 0.

$$\hat{P} = P_0 - \frac{(P_1 - P_0)^2}{(P_2 - 2P_1 + P_0)}$$

Substitute all p values with the existing values on the table.

$$\hat{P} = 1.5 - \frac{(1.348399725-1.5)^{2}}{(1.367376372-2(1.348399725)+1.5)}$$

Then, insert the value calculated into the table. To calculate p1 and p2 for k=1, use the same iteration method we did above in Step 5. Repeat step 6 again to get the next p0 value for k=2 and k=3.

k	p_0^k	p_1^k	p_2^k
0	1.5	1.348399725	1.367376372
1	1.365265224	1.365225534	1.365230583
2	1.365230013		
3			

$$\widehat{p_0^2} = 1.365230013$$

Example 7

Use **Steffensen's method** to find x3 – x – 1 = 0, to an accuracy of the root of that lies in [1, 2], Fill in the table by repeating the steps.

k	p_0^k	p_1^k	p_2^k	$\lvert P_n - P_{n-1}\rvert$
0	1	1.414213	1.306604	-
1	1.328777	1.323849	1.324905	0.328777
2	1.324719			0.0004058
3				

Use **Steffensen's method** to find $x - 2^{-x} = 0$, to an accuracy of the root of that lies in [0, 1], and compare this to the result of Secant method.

Steffensen's Method

$$x - 2^{-x} = 0$$

$$x = 2^{-x}$$

$$g(x) = 2^{-x}$$

$$p_0 = 0$$

$$\hat{p} = p_0 - \frac{(p_1 - p_0)^2}{p_2 - 2p_1 + p_0}$$

n	p_0	p_1	p_2	$\lvert p_n - p_{n-1} \rvert$
0	0.00000000	1.00000000	0.50000000	-
1	0.66666667	0.62996052	0.64619410	0.66666667
2	0.64121619	0.64117221	0.64119176	0.02545047
3	0.64118574	-	-	0.00003045