2.4

Error Analysis for Iterative Methods

1. Order of Convergence of a Sequence

- **Definition**: Suppose the sequence

  $p_0, p_1, p_2, \ldots, p_n, \ldots \boxed{->}$ p ← A constant

  Approaches

If λ>0 and α>0 exist with

$$\lim_{n->\infty} \frac{|p_{n+1} \boxed{- p}|}{|p_n \boxed{- p}|^{\alpha}} = \lambda$$

α

then the sequence $p_0, p_1, \ldots, p_n, \ldots$ converges to p of order α.

Order of Convergence

- **Definition**: λ is called asymptotic error constant.
- Important cases:
  - If $\alpha = 1$ and $0<\lambda\leq 1$, then the sequence is linearly convergent. degree 1
  - If $\alpha = 2$, then the sequence is quadratically convergent.
- Example: 1. Show that the sequence $p = \frac{1}{n^2}$ converges to 0 $(p=0)$ linearly.

$p=0$

$$\frac{\left|\frac{1}{(n+1)^2} - 0\right|}{\left|\frac{1}{n^2} - 0\right|^{1}} = \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2+2n+1} = 1$$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}}$$

$$\lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^2} - 0\right|}{\left|\frac{1}{n^2} - 0\right|^1} \qquad \frac{1}{n+1} \times n^2$$

$$\lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| \qquad \frac{1}{\infty} = 0$$

$$\lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| \qquad \frac{2}{\infty} = 0 \qquad \frac{1}{\infty} = \frac{1}{\infty} = 0$$

$$\lim_{n\to\infty} \left|\frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}}\right| = \lim_{n\to\infty} \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}} = \frac{1}{1} = 1$$

$$p_n = \frac{1}{n^2} \qquad p = 0 \qquad \alpha = 1$$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^2} - 0\right|}{\left|\frac{1}{n^2} - 0\right|^1} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2$$

$$= \lim_{n\to\infty} \left(\frac{\cancel{n}/\cancel{n}}{\cancel{n}/\cancel{n} + \frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)$$

$$= \frac{1}{1+\cancel{\frac{1}{\infty}}^{0}} = \frac{1}{1} = \boxed{1}$$ since it equals 1 $p_n \to p$ linearly

converges

$p_n = \frac{1}{n^3}$, does this sequence converges linearly?

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^3} - 0\right|}{\left|\frac{1}{n^3} - 0\right|^1} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^3}\right|}{\left|\frac{1}{n^3}\right|} = \lim_{n\to\infty} \left|\frac{n^3}{(n+1)^3}\right| = \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^3\right| = \lim_{n\to\infty} \left|\left(\frac{\cancel{n}/\cancel{n}}{\cancel{n}/\cancel{n} + \cancel{1/n}^{0}}\right)^3\right| = \lim_{n\to\infty} \left|\left(\frac{1}{1+0}\right)^3\right|$$

$$= \left|\left(\frac{1}{1}\right)^3\right| = 1$$ so converges

- 2. Show that the sequence $\boxed{p_n = 10^{-2^n}}$ converges to zero quadratically.

$p_{n+1} = 10^{-2^{n+1}}$

$10^{-2^n} = \frac{1}{10^{2^n}}$ so pos num

$p = 0$

$$\frac{|p_{n+1} - 0|}{|p_n - 0|^{\alpha}} = \frac{10^{-2^{n+1}}}{10^{-2*2^n}} = \frac{10^{-2*2^n}}{10^{-2*2^n}} = 1$$

$$\lim_{n\to\infty} \frac{|10^{-2^{n+1}} - 0|}{|10^{-2^n} - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^n \cdot 2}} = \lim_{n\to\infty} \frac{\cancel{10^{-2^{n+1}}}}{\cancel{10^{-2^{n+1}}}} = 1$$

p_n converges to p quadratically

2. Show that the sequence $p_n = 10^{-2^n}$ converges to zero quadratically.

$P_{n+1} = 10^{-2^{n+1}}$

$p = 0$

$\alpha = 2$

$\frac{n}{n}$

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lim_{n \to \infty} \frac{|10^{-2^{n+1}} - 0|}{|10^{-2^n} - 0|^{2}} = \lim_{n \to \infty} \frac{|\cancel{10^{-2^{n+1}}}|}{|\cancel{10^{-2^{n+1}}}|} = \boxed{1}$$

$$(10^{-2^n})^2$$

$$10^{-2^n \cdot 2^1} = 10^{-2^{n+1}}$$

$$10^{-4^n}$$

Is the sequence $\boxed{p_n = \frac{1}{n^3}}$ converges to 0 quadratically?

$p = 0$

$\alpha = 2$

$\boxed{p_{n+1} = \frac{1}{(n+1)^3}}$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}}$$

$$\lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^3} - 0\right|}{\left|\frac{1}{n^3} - 0\right|^{2}} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^3}\right|}{\left|\left(\frac{1}{n^3}\right)^2\right|}$$

$$= \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^3}\right|}{\left|\frac{1}{n^6}\right|}$$

$$= \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)^3}}{\frac{1}{n^6}}\right| = \lim_{n\to\infty} \left|\frac{n^6/n^3}{\frac{(n+1)^3}{n^3}}\right| = \lim_{n\to\infty} \frac{n^3}{\left(\frac{n+1}{n}\right)^3}$$

both have power 3

$$\lim_{n\to\infty} \frac{\frac{n^6}{n^3}}{\left(\frac{n}{n} + \frac{1}{n}\right)^3} = \frac{\frac{n^6}{n^3}}{\frac{n^3}{n^3} + \frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \left|\frac{n^6/n^3}{\frac{n^3}{n^3} + \frac{1}{n^3}}\right|$$

divide by the highest power of the denominator

$$= \lim_{n\to\infty} |n^3| = \infty$$

$$\frac{\frac{n^6}{n^3}}{\frac{n^3}{n^3} + \frac{1}{n^3}} = \frac{n^3}{1}$$

$$\lim_{n\to\infty} n^3 = \infty^3 = \infty$$

Does not converge quadratically.

however it converges linearly)

2.4 Exercise-7(a)

Examples:

1. Let $x_n = \frac{1}{n^k}$ for some fixed $k > 0$. Then we know this sequence converges to 0. Compute the rate of convergence:

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^\alpha} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^{\alpha k}}} = \lim_{n\to\infty} \left(\frac{n^\alpha}{n+1}\right)^k$$

We get convergence for $\alpha = 1$, but not $\alpha = 2$, so this sequence converges **linearly** to 0.

This is interesting- Sequences like the following:

$$x_n = \frac{1}{n}, \quad x_n = \frac{1}{n^2}, \quad x_n = \frac{1}{n^3}, \ldots$$

all converge **linearly** to zero.

2.4 Exercise-8(a)

2. Let $x_n = 10^{-2^n}$ (which converges to 0). Compute the rate of convergence:

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^\alpha} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^n\cdot\alpha}}$$

It's easy to see that if $\alpha = 2$, we get convergence.

2.4 Exercise-8(b)

Show that the sequence $p_n = 10^{-n^k}$ does not converge quadratically, regardless of the size of the exponent k.

Since

$$\frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \frac{10^{-(n+1)^k}}{10^{-2n^k}} = 10^{2n^k-(n+1)^k} \to \infty$$

as $n \to \infty$, we have that p_n does not converge quadratically to 0, for any positive integer k.

Exercise 2.4

1. Use Newton's method to find solutions accurate to within 10^{-5} to the following problems.

a. $x^2 - 2xe^{-x} + e^{-2x} = 0, \quad \text{for } 0 \leq x \leq 1$

Modified Newton's Method

Newton's method: $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

or $P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$ The same

Given an initial approximation p_0, generate a sequence $\{p_n\}_{n=0}^{\infty}$ by

Modefied Newtons method (will be given in the test)

$$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})}, \quad \text{for } n \geq 1.$$

Note: The modified Newton' method requires the second-order derivative $f''(x)$.

- Use Modified Newton's method to find solutions accurate to within 10-2 to the following problem.

- 2xcos2x -(x-2)2 = 0 , for [3, 4] $P_0 = 3.5$

$$f(x) = 2x\cos 2x - (x-2)^2 = 0$$

$$f'(x) = 2\cos 2x + 2x(-\sin 2x)\cdot 2 - 2(x-2) = \boxed{2\cos 2x - 4x\sin 2x - 2x + 4}$$

$$f''(x) = 2(-\sin 2x)\cdot 2 - 4\sin 2x - 4x(\cos 2x)\cdot 2 - 2 = -4\sin 2x - 4\sin 2x - 8x\cos 2x - 2 = \boxed{-8\sin 2x - 8\cos 2x - 2}$$

$$P_n = P_{n-1} - \frac{f(P_{n-1})\, f'(P_{n-1})}{[f'(P_{n-1})]^2 - f(P_{n-1})\, f''(P_{n-1})} \qquad P_1 = 3.5 - \frac{(-3.02715)(-10.690007)}{(-10.690007)^2 (3.02715)(-28.365156)} = 3.661692$$

$$f(P_0) = f(3.5) = 2(3.5)\cos(2(3.5)) - (3.5-2)^2 = 3.027315$$

$$f'(3.5) = 2\cos(2(3.5)) - 4(3.5)\sin(2(3.5)) - 2(3.5) + 4 = -10.690007$$

$$f''(3.5) = -8\sin(2(3.5)) - 8(3.5)\cos(2(3.5)) - 2 = -28.365156$$

$P_1 = 3.661692$

$f(P_1) = 2(P_1)\cos(2(P_1)) - (P_1 - 2)^2 =$

$f'(P_1) = 2\cos(2(P_1)) - 4(P_1)\sin(2(P_1)) - 2(P_1) + 4 =$

$f''(P_1) = -8\sin(2(P_1)) - 8(P_1)\cos(2(P_1)) - 2 =$