Chapter 2.3

Newton Raphson Method

Newton Raphson Method is an iterative numerical method used to find roots (solutions) of a real-valued function.

The method starts with an initial guess and uses calculus, specifically derivatives, to improve the accuracy of the solution with each iteration.

This concept is widely used in root-finding algorithms, numerical analysis, and engineering problem-solving.

Formula Used in Newton Raphson Method

The standard formula is: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ where:

- x_n is the current approximation
- $f(x)$ is the function whose root is being determined
- $f'(x)$ is the derivative of the function

Repeat the process until satisfactory accuracy is reached.

Worked Example – Solving a Problem

Example

Find a root of the equation $x^3 - 2x - 5 = 0$ using the Newton Raphson Method.

Solution

Start with $x_0 = 2$

1. Set $f(x) = x^3 - 2x - 5$, $f'(x) = 3x^2 - 2$, $x_0 = 2$.

2. Calculate $f(2) = 8 - 4 - 5 = -1$ and $f'(2) = 12 - 2 = 10$.

3. Apply the formula:

$$x_1 = 2 - \frac{-1}{10} = 2 + 0.1 = 2.1$$

4. Next, $f(2.1) = (2.1)^3 - 2 \times 2.1 - 5 = 9.261 - 4.2 - 5 = 0.061$

$$f'(2.1) = 3 \times (2.1)^2 - 2 = 3 \times 4.41 - 2 = 13.23 - 2 = 11.23$$

Solution cont'd

5. Update:

$$x_2 = 2.1 - \frac{0.061}{11.23} \approx 2.1 - 0.0054 \approx 2.0946$$

6. Repeat with x_2 for higher accuracy as needed.

Thus, the root is approximately **2.0946**.

Secant Method

A Secant Method refers to an iterative technique used in numerical analysis to find approximations to the roots of a real-valued function.

The secant method requires two initial guesses for the root and generates a sequence closer to the actual solution by drawing a secant line between two points on the function.

Formula Used in Secant Method

$\Rightarrow$ $f(x_n)$ will be given along with x_{n-1} and x_n

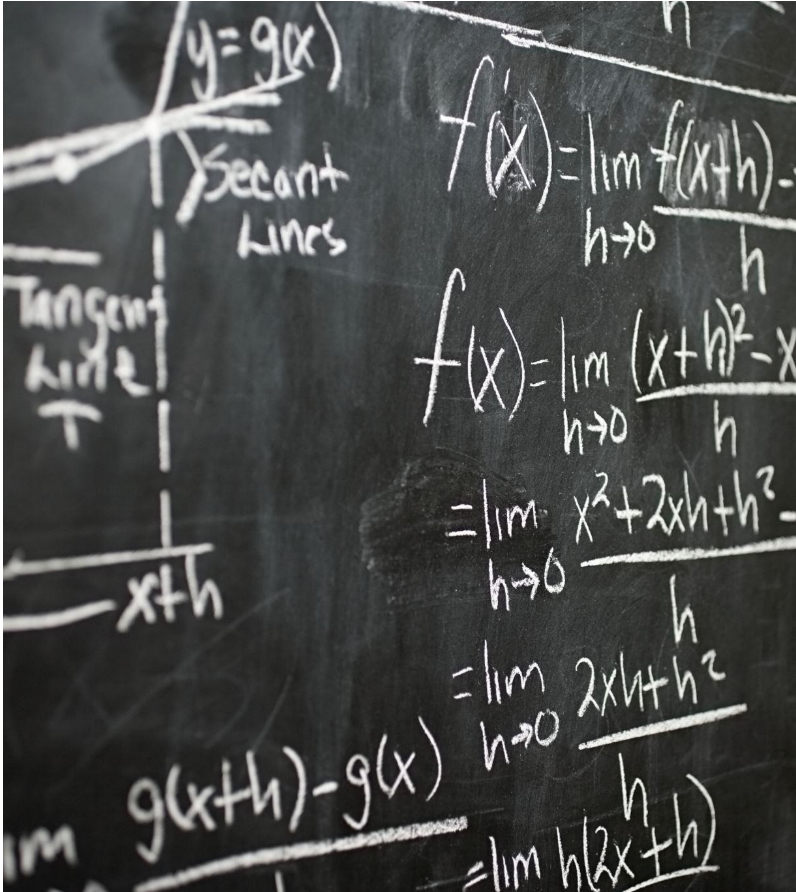
The standard formula is

$$\boxed{x_{n+1} = x_n - f(x_n) \times \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}}$$

Here, $\mathbf{x_n}$ and $\mathbf{x_{n-1}}$ are two previous approximations, and is the given function whose root is being found. This iterative formula does not require the explicit calculation of derivatives, unlike the Newton-Raphson method.

False Position Method

The False Position Method, also called Regula Falsi Method, is a numerical method to solve equations of the form . It is most useful when algebraic solutions are difficult or impossible. It is particularly popular for quickly finding the real root of nonlinear equations.



Formula Used in False Position Method

A combination of the secant method formula + Checking like bisectional method (first 2 values are opposite in sign)
Also check sign for each value

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

The standard formula is: $c = b - \dfrac{f(b)(b-a)}{f(b)-f(a)}$

EXERCISE SET 2.3

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?
3. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.
   - **c.** Which of **a.** or **b.** is closer to $\sqrt{6}$?
4. Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.
5. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.
   - **a.** $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$
   - **b.** $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$
   - **c.** $x - \cos x = 0, \quad [0, \pi/2]$
   - **d.** $x - 0.8 - 0.2 \sin x = 0, \quad [0, \pi/2]$
6. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.
   - **a.** $e^x + 2^{-x} + 2\cos x - 6 = 0$ for $1 \le x \le 2$
   - **b.** $\ln(x-1) + \cos(x-1) = 0$ for $1.3 \le x \le 2$
   - **c.** $2x\cos 2x - (x-2)^2 = 0$ for $2 \le x \le 3$ and $3 \le x \le 4$
   - **d.** $(x-2)^2 - \ln x = 0$ for $1 \le x \le 2$ and $e \le x \le 4$
   - **e.** $e^x - 3x^2 = 0$ for $0 \le x \le 1$ and $3 \le x \le 5$
   - **f.** $\sin x - e^{-x} = 0$ for $0 \le x \le 1$ $\;3 \le x \le 4$ and $6 \le x \le 7$
7. Repeat Exercise 5 using the Secant method.
8. Repeat Exercise 6 using the Secant method.
9. Repeat Exercise 5 using the method of False Position.

7) a) secant method

$x^3 - 2x^2 - 5 = 0$ $\quad [\overset{x_0}{1}, \overset{x_1}{4}]$ $\quad x_{n-1} = 1 \quad x_n = 4$

$f(1) = 1^3 - 2(1)^2 - 5 = -6$

$f(4) = 4^3 - 2(4)^2 - 5 = 27$

$x_{n+1} = x_n - f(x_n) \times \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

$4 - 27 \times \frac{4-1}{}$

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 4 - \frac{27(4-1)}{27 - -6} = 1.545455$$

$f(x_2) = (1.545455)^3 - 2(1.545455)^2 - 5 = -6.08565$

$$x_3 = 1.545455 - \frac{-6.08565(1.545455-4)}{-6.08565 - 27} = 1.996935$$

$f(x_3) = (1.996935)^3 - 2(1.996935)^2 - 5 = -5.012222$

$$x_4 = 1.996935 - \frac{(-5.012222)(1.996935 - 1.545455)}{-5.012222 - (-6.08565)}$$

d) $f(x) = x - 0.8 - 0.2 \sin x \quad [0, \frac{\pi}{2}]$

$f(0) = 0 - 0.8 - 0.2 \sin 0 = -0.8$

$f(\frac{\pi}{2}) = \frac{\pi}{2} - 0.8 - 0.2 \sin \frac{\pi}{2} = 0.570796$

$$x_2 = \frac{\pi}{2} - \frac{0.570796(\frac{\pi}{2} - 0)}{0.570796 - (-0.8)} = 0.916721$$

$f(x_2) = 0.916721 - 0.8 - 0.2 \sin 0.916721 = -0.042001$

$$x_3 = 0.916721 - \frac{-(0.042001)(0.916721 - \frac{\pi}{2})}{-0.042001 - 0.570796} = 0.961551$$

$f(x_3) = 0.961551 - 0.8 - 0.2 \sin 0.961551 = -0.002465$

$$x_4 = 0.961551 - \frac{(-0.002465)(0.961551 - 0.916721)}{-0.002465 - (-0.042001)} = \underline{0.964346}$$

$f(x_4) = x_4 - 0.8 - 0.2 \sin x_4 = 0.000011$

$$x_5 = 0.964346 - \frac{0.000011(0.964346 - 0.961551)}{0.000011 - (-0.002465)} = \underline{0.964334}$$

$|x_5 - x_4| = |0.964334 - 0.964346| = 0.000012 < 10^{-4}$

$x_5 = 0.964334$ is the approximate solution

$$x_5 = 0.739084$$

$$|x_5 - x_4| = |0.739084 - 0.739569| = 0.00048 > 10^{-4}$$

0.0001

$$f(x_5) = 0.739084 - \cos(0.739084) = -0.000002$$

The approx solution

$$x_6 = \underline{0.739084} - \frac{-0.000002\,(0.739084 - 0.739569)}{-0.000002 - 0.000809}$$

$$= 0.\underline{739085}$$

$$|x_6 - x_5| = |0.739085 - 0.739084|$$

$$= 0.000001 < 10^{-4}$$

Use false position method to find approx solution to 10^{-4}

9) b) $f(x)\ x^3+3x^2-1 \quad [\underset{a}{\overset{x_0}{-3}}, \underset{b}{\overset{x_1}{-2}}]$

$f(-3) = (-3)^3 + 3(-3)^2 - 1 = \boxed{-1}$

$f(-2) = (-2)^3 + 3(-2)^2 - 1 = \boxed{3}$

opp in sign so this method can be used

$x_2 = -2 - \dfrac{3(-2--3)}{3--1} = -2.75 \qquad [\underset{x_0}{\overset{a}{-3}}, \underset{x_1}{\overset{b}{-2.75}}]$

$f(-2.75) = (-2.75)^3 + 3(-2.75)^2 - 1 = 0.890625$ (+ve)

$x_3 = -2.75 - \dfrac{0.890625(-2.75--3)}{0.890625-(-1)} = \boxed{-2.87} \qquad [\underset{x_0}{-3}, \underset{x_1}{-2.75}]$

$f(x_3) = (-2.87)^3 + 3(-2.87)^2 - 1 = 0.070797$ $\boxed{(+ve)}$

$x_4 = -2.87 - \dfrac{0.070797(-2.87-(-3))}{0.070797-(-1)} = -2.878595$

$f(x_4) = (-2.878595)^3 + 3(-2.878595)^2 - 1 = 0.005999$ (+ve) $\quad [-3, -2.878595]$

$x_5 = -2.87895 - \dfrac{0.005999(-2.878595-(-3))}{0.005999-(-1)} = -2.879319 \qquad |x_5 - x_4| = |-2.879319 - (-2.87895)| = 0.0007 < 10^{-4}$

$f(x_5) = (-2.879319)^3 + 3(-2.879319)^2 - 1 = 0.000503163 \quad [-3, -2.879319]$

$x_6 = -2.879319 - \dfrac{0.000503163(-2.879319-(-3))}{0.000503163-(-1)} = -2.879319$

$|x_6 - x_5| = |-2.879319 - (-2.879319)| = 0.00009 < 10^{-4}$

Ch-2

Finding approx. sol^n of a fu^n

- 2.1 Bisection
- 2.2 Fixed point
- 2.3
  - Newton's
  - Secant
  - False-position

Newton's

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Secant, False-position: Same formula

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

False-position → Use the checking as you did in bisection method: $f(x_n)$ and $f(x_{n-1})$ are opposite in sign

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$[0, 1]$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$c = b - \frac{f(b)(b-a)}{f(b) - f(a)}$$

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$