2.1

Bisection Method

* IVT: if f(a) and f(b) are opposites in sign, then there is a $c \in [a,b]$, such that $f(c)=0$ $f(a)=+ve \rightarrow f(b)=-ve$

* Finding the approximate value of x for which $f(x)=y=0$

* Finding approximate solution of a function * We try to decrease the interval by dividing it by 2 each time till we find c

Bisection Technique The first technique, based on the Intermediate Value Theorem, is called the Bisection, or Binary-search, method. Suppose /is a continuous function defined on the interval [a, b], with f(a) and f(b) of opposite sign.

The Intermediate Value Theorem implies that a number p exists in (a, b) with f(p) = 0.

Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique. The method calls for a repeated halving (or bisecting) of subintervals of [a, b] and, at each step, locating the half containing p.

To begin, set a1= a and b1= b and let p1 be the midpoint of [a, b], that is.

P1 = a1 + $\frac{b_1 - a_1}{2}$

If f{p1) = 0, then p = p1, and we are done.

If f(P1) ≠ 0, then f(pi) has the same sign as either f(a1) or f(b1).

o If f(p1) and f(a1) have the same sign, p ε (p1 ,b1). Set a2 = p1 and b2 = b1.

o If f{p1) and f(a1) have opposite signs, p (a1, p1). Set a2 = a1 and b2 = p1. Then reapply the process to the interval [a2,b2]

Bisectional method steps:

① check $f(a)$, $f(b)$ [opposite in sign]

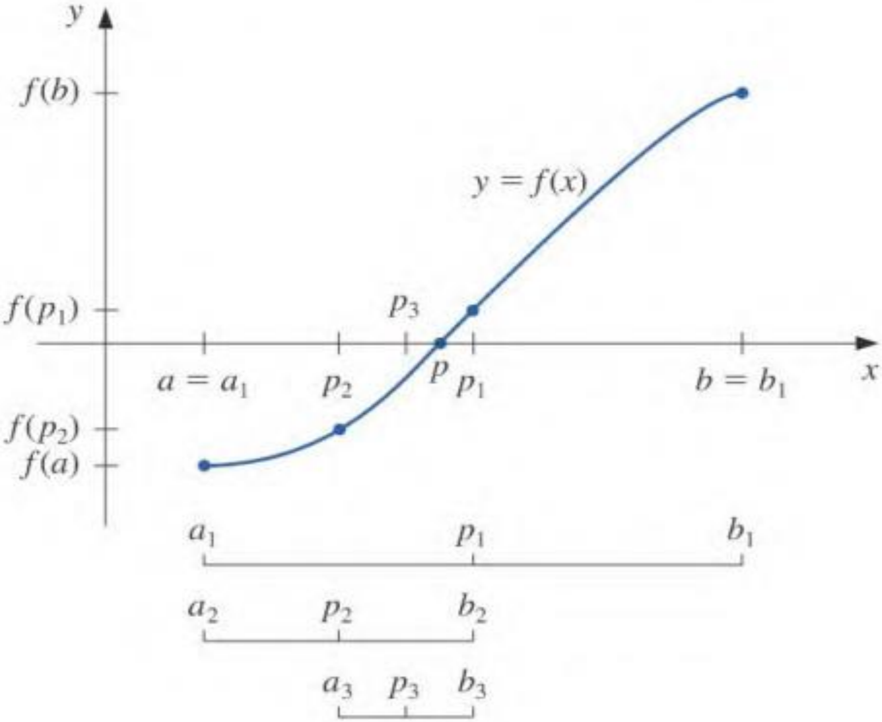
② $P_1 = \frac{a-b}{2}$

③ $f(P_1) \rightarrow$ check the sign

* Sign of $f(P_1)$ is the same sign of $f(a)$ then replace a by P_1 $P_2 = \frac{P_1+b}{2}$,new interval $[P_1, b]$

* else if the sign of $f(P_1)$ is the same as sign of $f(b)$ then replace b by P_1 $P_2 = \frac{a+P_1}{2}$, new interval $[a, P_1]$

④ $f(P_2)$ check the sign and do the same as step 3 till the P number given to stop



y
f(b)
y = f(x)
f(p1)
p3
a = a1
p2
p
p1
b = b1
x
f(p2)
f(a)
a1
p1
b1
a2
p2
b2
a3
p3
b3

EXERCISE SET 2.1

1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x = 0$ on $[0, 1]$.
2. Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1) = 0$. Use the Bisection method on the following intervals to find p_3.
   a. $[-2, 1.5]$ b. $[-1.25, 2.5]$
3. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.
   a. $[0, 1]$ b. $[1, 3.2]$ c. $[3.2, 4]$
4. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on each interval.
   a. $[-2, -1]$ b. $[0, 2]$ c. $[2, 3]$ d. $[-1, 0]$
5. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.
   a. $x - 2^{-x} = 0$ for $0 \le x \le 1$
   b. $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$
   c. $2x\cos(2x) - (x+1)^2 = 0$ for $-3 \le x \le -2$ and $-1 \le x \le 0$
   d. $x\cos x - 2x^2 + 3x - 1 = 0$ for $0.2 \le x \le 0.3$ and $1.2 \le x \le 1.3$
6. Use the Bisection method to find solutions, accurate to within 10^{-5} for the following problems.
   a. $3x - e^x = 0$ for $1 \le x \le 2$
   b. $2x + 3\cos x - e^x = 0$ for $0 \le x \le 1$
   c. $x^2 - 4x + 4 - \ln x = 0$ for $1 \le x \le 2$ and $2 \le x \le 4$
   d. $x + 1 - 2\sin \pi x = 0$ for $0 \le x \le 0.5$ and $0.5 \le x \le 1$
7. a. Sketch the graphs of $y = x$ and $y = 2\sin x$.
   b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2\sin x$.
8. a. Sketch the graphs of $y = x$ and $y = \tan x$.
   b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = \tan x$.
9. a. Sketch the graphs of $y = e^x - 2$ and $y = \cos(e^x - 2)$.

① $f(x) = \sqrt{x} - \cos x = 0$ $\quad [\overset{a}{0}, \overset{b}{1}]$

$f(a) = f(0) = \sqrt{0} - \cos 0 = -ve$

$f(b) = f(1) = \sqrt{1} - \cos 1 = +ve$

By IVT, there is a $c \in [0,1]$ such that $f(c)=0$

$P_1 = \frac{0+1}{2} = \frac{1}{2} = 0.5$ (same sign as b)

$f(0.5) = \sqrt{0.5} - \cos(0.5) = -ve$

Replace a by P_1 $[0.5, 1]$

$P_2 = \frac{0.5+1}{2} = \frac{3}{4} = 0.75$ (same sign as b)

$f(0.75) = \sqrt{0.75} - \cos(0.75) = +ve$ $[0.5, 0.75]$

$P_3 = \frac{0.5+0.75}{2} = \frac{5}{8} = 0.625$

If we need to find P_4

$f(0.625) = \sqrt{0.625} - \cos 0.625 = -ve$ $[0.625, 0.75]$

$P_4 = \frac{0.625+0.75}{2} = \frac{11}{16} = 0.6875$

③ $f(x) = x^3 - 7x^2 + 14x - 6 = 0$

P_1, P_2, P_3, P_4

$(P_3 - P_2) \leq 10^{-2}$

$(P_4 - P_3) \leq$

ⓒ $[3.2, 4]$

$f(3.2) = (3.2)^3 - 7(3.2)^2 + 14(3.2) - 6 = -ve$

$f(4) = (4)^3 - 7(4)^2 + 14(4) - 6 = +ve$

By IVT, there is a $c \in [3.2, 4]$ such that $f(c)=0$

$P_1 = \frac{3.2+4}{2} = 3.6$

$f(3.6) = (3.6)^3 - 7(3.6)^2 + 14(3.6) - 6 = +ve$ $[3.2, 3.6]$

$P_2 = \frac{3.2+3.6}{2} = 3.4$

$f(3.4) = (3.4)^3 - 7(3.4)^2 + 14(3.4) - 6 = +ve$ $[3.4, 3.6]$

$P_3 = \frac{3.2+3.6}{2} = 3.5$

$f(3.5) = (3.5)^3 - 7(3.5)^2 + 14(3.5) - 6 = +ve$ $[3.4, 3.5]$

$P_4 = \frac{3.4+3.5}{2} = 3.45$ $\quad |P_4 - P_3| = 0.05$

$f(3.45) = +ve$ $[3.4, 3.45]$ $\quad |P_5 - P_4| = 0.025$

$P_5 = \frac{3.4+3.45}{2} = 3.425$

$f(3.425) = +ve$ $[3.4, 3.425]$ $\quad |P_6 - P_5| = 0.0125$

$P_6 = \frac{3.425+3.4}{2} = 3.4125$

$f(3.4125) = -ve$ $[3.4125, 3.425]$ $\quad |P_7 - P_6| = 0.00625 \leq 10^{-2}$

$P_7 = \frac{3.4125+3.425}{2} = 3.41875$ $\quad P_7$ is the approximate solution

5. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.

a. $x - 2^{-x} = 0$ for $0 \le x \le 1$ (upto p3)

$f(0) = 0 - 2^{-0} = -1$ (-ve)   [-ve, +ve]

$f(1) = 1 - 2^{-1} = \frac{1}{2}$ (+ve)

$P_1 = \frac{0+1}{2} = \frac{1}{2} = 0.5$

$f(\frac{1}{2}) = \frac{1}{2} - 2^{-\frac{1}{2}} = -0.2$ (-ve) [0.5, 1]

$P_2 = \frac{\frac{1}{2}+1}{2} = \frac{3}{4} = 0.75$

$f(\frac{3}{4}) = \frac{3}{4} - 2^{-\frac{3}{4}} = 0.155$ (+ve) [0.5, 0.75]

$P_3 = \frac{\frac{1}{2}+\frac{3}{4}}{2} = \frac{5}{8} = 0.625$

b. $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$ upto 7th iteration ($P_7 = 0.2578125$)

upto 10^{-2}

$f(0) = -1$ (-ve) [-ve, +ve]

$f(1) = 2.7$ (+ve)

$P_1 = \frac{0+1}{2} = 0.5$

$f(0.5) = 0.9$ (+ve) $[0, 0.5]$

$P_2 = \frac{0+0.5}{2} = 0.25$

$f(0.25) = -0.03$ (-ve) $[0.25, 0.5]$

$P_3 = \frac{0.25+0.5}{2} = 0.375$

$f(0.375) = 0.44$ (+ve) $[0.25, 0.375]$

$P_4 = \frac{0.25+0.375}{2} = 0.3125$

$f(0.3125) = 0.2$ (+ve) $[0.25, 0.3125]$

$P_5 = \frac{0.3125+0.25}{2} = 0.28125$

$f(0.28125) = 0.32$ (+ve) $[0.25, 0.28125]$

$P_6 = \frac{0.25+0.28125}{2} = 0.265725$

$f(0.265725) =$ +ve $[0.25, 0.265725]$

$P_7 = \frac{0.25+0.265725}{2} = 0.2578125$

$|P_4 - P_3| = |0.3125 - 0.375| = 0.0625$

$|P_5 - P_4| = |0.28125 - 0.3125| = 0.03125$

$|P_6 - P_5| = |0.265625 - 0.28125| = 0.015$

$|P_7 - P_6| = 0.2578125 - 0.265625| = 0.0078$

The process we used is called iteration