Numerical Integration

① Closed Cotes :

→ Trapezoidal rule $n=1$:

$$\int_{a=x_0}^{x_1=b} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] \quad , \; h = x_1 - x_0$$

→ Simpson's rule $n=2$:

$$\int_{a=x_0}^{x_2=b} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(\underset{\frac{a+b}{2}}{x_1}) + f(x_2)\right] \quad , \; h = \frac{b-a}{2}$$

→ Simpson's $3^{th}/8$ rule $n=3$:

✗ $$\int_{a=x_0}^{x_3=b} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] \quad , \; h = \frac{b-a}{2}$$

✗ → $n=4$ ('ll be given, and asked in Q use $n=4$ formula)

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right] \quad , \; h = \frac{b-a}{2}$$

Q1 **1.** Approximate the following integrals using the Trapezoidal rule.

h. $\int_{x_0=0}^{x_1=\pi/4} \underbrace{e^{3x} \sin 2x}_{f(x)} \, dx$

Trapezoidal rule : $\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)]$

$h = x_1 - x_0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

$$\int_0^{\pi/4} e^{3x} \sin 2x \, dx = \frac{\frac{\pi}{4}}{2}\left[e^{3(0)} \sin 2(0) + e^{3(\pi/4)} \sin 2(\pi/4)\right]$$

$$= 4.143259655$$

a. $\int_{0.5}^{1} x^4 \, dx$ $\qquad \int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)]$

$$\int_{0.5}^{1} x^4 \, dx = \frac{0.5}{2}\left[(0.5)^4 + (1)^4\right] = 0.265625$$

$h = 1 - 0.5 = 0.5$

Q5 **5.** Repeat Exercise 1 using Simpson's rule. $\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$

c. $\int_{1}^{1.5} x^2 \ln x \, dx$ $\qquad h = \frac{1.5 - 1}{2} = 0.25$

$$\int_{x_0=1}^{x_2=1.5} x^2 \ln x \, dx = \frac{0.25}{3}\left[f(1) + 4\underbrace{f(1.25)}_{x_1 = \frac{1+1.5}{2} = 1.25} + f(1.5)\right] = 0.19224530$$

② Open Cotes :

"in Close Cote the Value of "n" will be given in Q"

$\rightarrow$ Midpoint Rule $(n=0)$ (NOT given)

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2h f(x_0) \qquad h = \frac{b-a}{n+2}$$

$\rightarrow$ $n=1$ ('ll be given)

$$\int_{x_{-1}}^{x_2 = x_{n+1}} f(x)\,dx = \frac{3h}{2}\left[f(x_0) + f(x_1)\right] \qquad h = \frac{b-a}{n+2}$$

$\rightarrow$ $n=2$

$$\int_{x_{-1}}^{x_3 = x_{n+1}} f(x)\,dx = \frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2)\right] \qquad h = \frac{b-a}{n+2}$$

$\rightarrow$ $n=3$

$$\int_{x_{-1}}^{x_4 = x_{n+1}} f(x)\,dx = \frac{5h}{24}\left[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)\right] \qquad h = \frac{b-a}{n+2}$$

Q9 :

9. Repeat Exercise 1 using the Midpoint rule.

c. $\int_1^{1.5} x^2 \ln x \, dx$ $\qquad h = \frac{b-a}{n+2} = \frac{1.5-1}{0+2} = 0.25$

$$\int_{x_{-1}}^{x_1} x^2 \ln x \, dx = 2h\, f(x_0)$$

$$\int_1^{1.5} x^2 \ln x \, dx = 2(0.25)\left(\underbrace{(1.25)^2 \ln(1.25)}_{x_0 = \frac{x_1 + x_{-1}}{2} = 1.25}\right) = 0.1743308$$

Q13 :

13. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 4, and Simpson's rule gives the value 2. What is $f(1)$?

Trapezoidal $\Rightarrow \int_0^2 f(x)\,dx = \frac{2}{2}[f(0) + f(2)] = 4$ $\qquad h = 2-0 = 2$

$\Rightarrow f(0) + f(2) = 4$

Simpson's $\Rightarrow \int_0^2 f(x)\,dx = 3\times\boxed{\frac{1}{3}}[f(0) + 4f(\overset{\frac{2+0}{2}=1}{1}) + f(2)] = 2 \times 3$ $\qquad h = \frac{2-0}{2} = 1$

$\Rightarrow f(0) + 4f(1) + f(2) = 6$

$$\Rightarrow \begin{array}{l} \cancel{f(0)} + 4f(1) + \cancel{f(2)} = 6 \\ - \quad \cancel{f(0)} + 0 + \cancel{f(2)} = 4 \\ \hline \frac{4f(1)}{4} = \frac{2}{4} = \frac{1}{2} \end{array}$$

$$\Rightarrow \boxed{f(1) = \frac{1}{2}}$$