Cubic Spline Interpolation

→ Types of Cubic spline :

Cubic Spline

Natural — Clamped

Natural: 2 values, 3 values

Clamped: 2 values, 3 values

① Construct the natural Cupic Spline :

"b" will given to can find A

⇒ if you have x_0, x_1 only → you'll find $S(x_0)$

⇒ if you " x_0, x_1, x_2 → you'll find $S(x_0)$ and $S(x_1)$

$S(x_i) = a_i + b_i(x - x_i) + c_i(x - x_i) + d_0(x - x_i)$

→ a_i and x_i from Q

→ $A^{-1} \cdot b \begin{bmatrix} * \\ * \end{bmatrix} \begin{matrix} \to c_0 \\ \to c_1 \end{matrix}$

→ $b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_2)$

→ $d_0 = \frac{1}{3h_0}(c_1 - C_0)$

→ $h_0 = x_1 - x_0$

② Clamped Spline :

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$;

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$;

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$; (*Implied by* **(b)**.)

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(f) One of the following sets of boundary conditions is satisfied:

(i) $S''(x_0) = S''(x_n) = 0$ (**natural** *(or free)* **boundary**);

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped boundary**).

① Natural Cubic Spline :

Q3 : **3.** Construct the $\boxed{\text{natural cubic}}$ spline for the following data.

a.

x	$f(x)$
$x_0 = 8.3$	$17.56492 = a_0$
$x_1 = 8.6$	$18.50515 = a_1$

→ 2value natural Cupic

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ "Identity matrix"

$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

given → $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A^{-1} \cdot b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} \to c_0 \\ \to c_1 \end{matrix}$

$h_0 = 8.6 - 8.3 = 0.3$

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0)$$

$$= \frac{1}{0.3}(18.50515 - 17.56492) - \frac{0.3}{3}(0 + 2(0)) = 3.1341$$

$$d_0 = \frac{1}{3h_0}(c_1 - c_0)$$

$$= \frac{1}{3(0.3)}(0 - 0) = 0$$

$$\Rightarrow S(x_0) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$

$$= 17.56492 + (3.1341)(x - 8.3) + 0\cancel{(x - 8.3)^2} + 0\cancel{(x - 8.3)^3}$$

$$= 17.56492 + (3.1341)(x - 8.3$$

b.	x	$f(x)$
$x_0 =$	0.8	0.22363362 $= a_0$
$x_1 =$	1.0	0.65809197 $= a_1$

$h_0 = 0.2$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{-1} \cdot b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} \to c_0 \\ \to c_1 \end{matrix}$$

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 - 2c_0)$$

$$= \frac{1}{0.2}(0.65809197 - 0.22363362) - \frac{0.2}{3}(0 - 2(0)) = 2.17229175$$

$$d_0 = \frac{1}{3h_0}(c_1 - c_0) = \frac{1}{3(0.2)}(0 - 0) = 0$$

$$S(x_0) = a_0 + b_0(x - x_0) + d_0(x - x_0)^2 + c_0(x - x_0)^3$$

$$= 0.22363362 + 2.17229175(x - 0.8) + 0\cancel{(x - 0.8)^2} + 0\cancel{(x - 0.8)^3}$$

$$= 0.22363362 + 2.17229175(x - 0.8)$$

c.

x	$f(x)$
$x_0 = -0.5$	$-0.0247500 \rightarrow a_0$
$x_1 = -0.25$	$0.3349375 \rightarrow a_1$
$x_2 = 0$	$1.1010000 \rightarrow a_2$

$h_0 = -0.25 - -0.5 = 0.25$, $h_1 = 0 + 0.25 = 0.25$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0.25 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -4.8765 \\ 0 \end{bmatrix}$$

$$A^{-1} \cdot b = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} \Rightarrow \quad A^{-1} = \xrightarrow{-0.25 R_1 + R_2} \begin{array}{c} \begin{array}{cc} A & \qquad I \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0.25 & 1 & 0.25 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0.25 & 0.25 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-0.25 R_3 + R_2 \rightarrow R_2} \begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.25 & 1 & -0.25 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cc} I & \qquad A^{-1} \end{array} \end{array}$$

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 - 2c_0)$$

$$= 1.845125$$

$$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 - 2c_1)$$

$$= 2.2515$$

$$A^{-1} \cdot B = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & -0.25 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -4.8765 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -4.8765 \\ 0 \end{bmatrix} \begin{array}{l} \rightarrow c_0 \\ \rightarrow c_1 \\ \rightarrow c_2 \end{array}$$

$$d_0 = \frac{1}{3h_0}(c_1 - c_0) = -6.502$$

$$d_1 = \frac{1}{3h_1}(c_2 - c_1) = 6.502$$

$$s(x_0) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 = -0.0247500 + 1.845125(x+0.5) + 0(x+0.5)^2 + -6.502(x+0.5)^3$$

$$s(x_1) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 = 0.3349375 + 2.2515(x+0.25) + -4.8765(x+0.25)^2 + 6.502(x+0.25)^3$$

② Clamped Cubic Spline :

must use these rulse :

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$;

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$;

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$; (*Implied by* **(b)**.)

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(f) One of the following sets of boundary conditions is satisfied:

(i) $S''(x_0) = S''(x_n) = 0$ (**natural** *(or free)* **boundary**);

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ **(clamped boundary)**.