Lagrange Intepolating Polynomials

formula :

$\rightarrow$ $L_0(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$

No maintioning x_0

just here

$\rightarrow$ $L_1(x) = \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$ Should match.

Polynomial $\rightarrow$ $P(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$

for find Absalute Erorr (Erorr Bound) :

① Approxmate.E $\Rightarrow$ by Subsituating $P(x)$ with x that given in Q

② Exact.E $\Rightarrow$ by Subsituating $f(x)$ with x that given in Q

③ find Absalute Erorr $\Rightarrow$ $|\text{Appr.E} - \text{Exact.E}|$

Q1:

1. For the given functions $f(x)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(\underline{0.45})$ and find the absolute error.

$\hookrightarrow x$

a. $f(x) = \cos x$

3 points $\Rightarrow$ $n=2$

Degree 1: $x_0=0$, $x_1 = 0.6$, $x_2 = 0.9$

① $f(x_0) = \cos(0) = 1$

$f(x_1) = 0.8253356$

② $L_0 = \dfrac{x - x_1}{x_0 - x_1} = \dfrac{x - 0.6}{0 - 0.6} = \dfrac{x - 0.6}{-0.6}$

$L_1 = \dfrac{x - x_0}{x_1 - x_0} = \dfrac{x - 0}{0.6 - 0} = \dfrac{x}{0.6}$

③ $p(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \dfrac{x-0.6}{-0.6}(1) + \dfrac{x}{0.6}(0.8253356)$

Absalute Error: $x = 0.45$

Appr.E $\Rightarrow$ $p(0.45) = \dfrac{0.45 - 0.6}{-0.6}(1) + \dfrac{0.45}{0.6}(0.8253356) = 0.8690017$

Exact.E $\Rightarrow$ $f(0.45) = \cos(0.45) = 0.9004471024$

Absalut.E $\Rightarrow$ $|\text{Appr.E} - \text{Exact.E}| = |0.8690017 - 0.9004471| = 0.0314454$

Degree 2: $x_0=0$, $x_1 = 0.6$, $x_2 = 0.9$

① $f(x_0) = \cos(0) = 1$

$f(x_1) = 0.8253356$

$f(x_2) = 0.6216099$

② $L_0 = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \dfrac{(x - 0.6)(x - 0.9)}{(0 - 0.6)(0 - 0.9)} = \dfrac{(x - 0.6)(x - 0.9)}{0.54}$

$L_1 = \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \dfrac{(x - 0)(x - 0.9)}{(0.6 - 0)(0.6 - 0.9)} = \dfrac{x(x - 0.9)}{-0.18}$

$$L_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-0.6)}{(0.9-0)(0.9-06)} = \frac{x(x-0.6)}{0.27}$$

③ $p(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$

$$= \frac{(x-06)(x-0.9)}{0.54}(1) + \frac{x(x-0.9)}{-0.18}(0.8253356) + \frac{x(x-0.6)}{0.27}(0.6216099)$$

Absalute Error : $x = 0.45$

Appr.E $\Rightarrow$ $p(0.45) = \frac{(0.45-06)(0.45-0.9)}{0.54}(1) + \frac{0.45(0.45-0.9)}{-0.18}(0.8253356) + \frac{0.45(0.45-0.6)}{0.27}(0.6216099)$

$p(0.45) = 0.898100075$

Exact.E $\Rightarrow$ $f(0.45) = \cos(0.45) = 0.9004471024$

Absalut.E $\Rightarrow$ $|\text{Appr.E} - \text{Exact.E}| = |0.898100075 - 0.9004471| = 0.0023$

So degree 2 has better approximaction value than degree 1 as well absalute error for degree 2 is less than degre 1

b. $f(x) = \sqrt{1+x}$ $x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$

Degree 1:

① $f(x_0) = \sqrt{1+0} = 1$

$f(x_1) = \sqrt{1+0.6} = 1.2649110$

② $L_0 = \frac{x-x_1}{x_0-x_1} = \frac{x-0.6}{0-0.6}$

$L_1 = \frac{x-x_0}{x_1-x_0} = \frac{x-0}{0.6-0}$

③ $p(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x-0.6}{0-0.6}(1) + \frac{x-0}{0.6-0}(1.2649110)$

Absalute Error : $x = 0.45$

$\Rightarrow p(0.45) = \frac{0.45-0.6}{0-0.6}(1) + \frac{0.45-0}{0.6-0}(1.2649110) = 1.1986832$

$\Rightarrow f(0.45) = \sqrt{1+0.45} = 1.2041594$

$\Rightarrow$ absalute.E $= |1.1986832 - 1.2041594| = 0.00547$

Degree 2 : $x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$

① $f(x_0) = \sqrt{1+0} = 1$

$f(x_1) = \sqrt{1+0.6} = 1.2649110$

$f(x_2) = \sqrt{1+0.9} = 1.3784048$

② $L_0 = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \dfrac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)} = \dfrac{(x-0.6)(x-0.9)}{0.54}$

$L_1 = \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \dfrac{(x-0)(x-0.6)}{(0.6-0)(0.6-0.9)} = \dfrac{x(x-0.6)}{-0.18}$

$L_2 = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \dfrac{(x-0)(x-0.6)}{(0.9-0)(0.9-0.6)} = \dfrac{x(x-0.6)}{0.27}$

③ $P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$

$$= \frac{(x-0.6)(x-0.9)}{0.54}(1) + \frac{x(x-0.6)}{-0.18}(1.2649110) + \frac{x(x-0.6)}{0.27}(1.3784048)$$

Absalute Error : $x = 0.45$

$$\Rightarrow P(0.45) = \frac{(0.45-0.6)(0.45-0.9)}{0.54} + \frac{0.45(0.45-0.6)}{-0.18}(1.2649110) + \frac{0.45(0.45-0.6)}{0.27}(1.3784048)$$

$= 1.203423728$

$\Rightarrow f(0.45) = \sqrt{1+0.45} = 1.2041594$

$\Rightarrow$ absulate. E $= |1.203423728 - 1.2041594| = 0.0007$

So degree 2 is better than degree 1

Q58 **5.** Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

$\Rightarrow$ find Lagrange and polynomails :

$n=3 \Rightarrow 4$ points

degree 1 : $x_0 = 8.1$, $x_1 = 8.3$, $x_2 = 8.6$, $x_3 = 8.7$

$$L_0 = \frac{(x-x_1)}{(x_0-x_1)} = \frac{(x-8.3)}{(8.1-8.3)} = \frac{(x-8.3)}{-0.2}$$

$$L_1 = \frac{(x-x_0)}{(x_1-x_0)} = \frac{(x-8.1)}{(8.3-8.1)} = \frac{(x-8.1)}{0.2}$$

$$P_1(x) = \frac{(x-8.3)}{-0.2}(16.94410) + \frac{(x-8.1)}{0.2}(17.56492)$$

degree 2 : $x_0 = 8.1$, $x_1 = 8.3$, $x_2 = 8.6$, $x_3 = 8.7$

$$L_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-8.1)(x-8.6)}{(8.1-8.3)(8.1-8.6)}$$

$$L_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-8.1)(x-8.6)}{(8.3-8.1)(8.3-8.6)}$$

$$L_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-8.1)(x-8.3)}{(8.6-8.1)(8.6-8.3)}$$

$$P_2(x) = \frac{(x-8.1)(x-8.6)}{(8.1-8.3)(8.1-8.6)}(16.94410) + \frac{(x-8.1)(x-8.6)}{(8.3-8.1)(8.3-8.6)}(17.56492) + \frac{(x-8.1)(x-8.3)}{(8.6-8.1)(8.6-8.3)}(18.50515)$$

degree 3 : $x_0 = 8.1$, $x_1 = 8.3$, $x_2 = 8.6$, $x_3 = 8.7$

$$L_0 = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-8.3)(x-8.6)(x-8.7)}{(8.1-8.3)(8.1-8.6)(8.1-8.7)}$$

$$L_1 = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-8.1)(x-8.6)(x-8.7)}{(8.3-8.1)(8.3-8.6)(8.3-8.7)}$$

$$L_2 = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-8.1)(x-8.3)(x-8.7)}{(8.6-8.1)(8.6-8.3)(8.6-8.7)}$$

$$L_3 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-8.1)(x-8.3)(x-8.6)}{(8.7-8.1)(8.7-8.3)(8.7-8.6)}$$

$$p(x) = \frac{(x-8.3)(x-8.6)(x-8.7)}{(8.1-8.3)(8.1-8.6)(8.1-8.7)}(16.94410) + \frac{(x-8.1)(x-8.6)(x-8.7)}{(8.3-8.1)(8.3-8.6)(8.3-8.7)}(17.56492) + \frac{(x-8.1)(x-8.3)(x-8.7)}{(8.6-8.1)(8.6-8.3)(8.6-8.7)}(18.50515) + \frac{(x-8.1)(x-8.3)(x-8.6)}{(8.7-8.1)(8.7-8.3)(8.7-8.6)}(18.82091)$$

Q7:

7. The data for Exercise 5 were generated using the following functions. Use the error formula to find a bound for the error and compare the bound to the actual error for the cases $n = 1$ and $n = 2$.

a. $f(x) = x \ln x$

from Q5 ⇒ **a.** $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

$$P_1(x) = \frac{(x-8.3)}{-0.2}(16.94410) + \frac{(x-8.1)}{0.2}(17.56492)$$

$$= \frac{(8.4-8.3)}{-0.2}(16.94410) + \frac{(8.4-8.1)}{0.2}(17.56492) = 17.87533$$

$$P_2(x) = \frac{(8.4-8.1)(8.4-8.6)}{(8.1-8.3)(8.1-8.6)}(16.94410) + \frac{(8.4-8.1)(8.4-8.6)}{(8.3-8.1)(8.3-8.6)}(17.56492) + \frac{(8.4-8.1)(8.4-8.3)}{(8.6-8.1)(8.6-8.3)}(18.50515)$$

$$= 17.87713$$

Exact. E $= 8.4 \ln(8.4) = 17.87714633$

$P_1(x) = |17.87533 - 17.87714633| = 0.001$

$P_2(x) = |17.87713 - 17.87714633| = 0.0001$