① Aitken's Method :

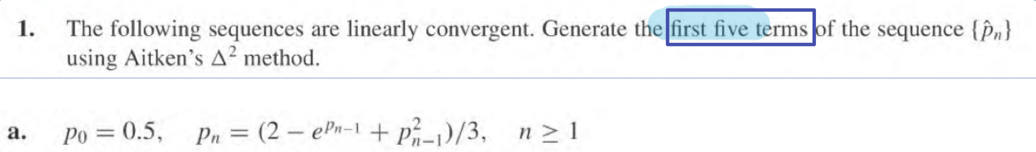
Formula :
$$\hat{P}_n = P_n - \frac{(P_{n+1} - P_n)^2}{P_{n+2} - 2P_{n+1} - P_n}$$

→ Alway Starts w/ P_0^1

→ $\hat{P}_n$ in formula P_0^{i} → Change depend on Iterations, → Same, P_0^2, P_0^3 ...

→ if P_n in $g(P_1^0)$ → remain the Same, → change depend on Iterations, P_2^0, P_3^0

Q1 :



1. The following sequences are linearly convergent. Generate the first five terms of the sequence $\{\hat{p}_n\}$ using Aitken's Δ^2 method.

a. $p_0 = 0.5, \quad p_n = (2 - e^{p_{n-1}} + p_{n-1}^2)/3, \quad n \geq 1$

if asked to find 5 terms we need to find 6 P's

⇒ apply P_n formula :

$P_0 = 0.5$

$$\Rightarrow P_1 = \frac{(2 - e^{P_0} + (P_0)^2)}{3} = 0.2004262$$

$$P_2 = \frac{(2 - e^{P_1} + (P_1)^2)}{3} = 0.2727490$$

$$P_3 = \frac{(2 - e^{P_2} + (P_2)^2)}{3} = 0.2536071$$

$P_4 = 0.2585503$

$P_5 = 0.2572656$

$P_6 = 0.2575989$

⇒ apply P_6 in Aitken's formula :

$$\hat{P}_n = P_n - \frac{(P_{n+1} - P_n)^2}{P_{n+2} - 2P_{n+1} - P_n}$$

⇒ Start's from $\hat{P}_0$

$$\hat{P}_0 = P_0 - \frac{(P_1 - P_0)^2}{P_2 - 2P_1 + P_0} = 0.5 - \frac{(0.2004262 - 0.5)^2}{0.2727490 - 2(0.2004262) + 0.5} = 0.2686843$$

$$\hat{P}_1 = P_1 - \frac{(P_2 - P_1)^2}{P_3 - 2P_1 + P_1} = 0.2576131$$

$$\hat{P}_2 = P_2 - \frac{(P_3 - P_2)^2}{P_4 - 2P_3 + P} = \underline{0.2575}058$$

$$\hat{P}_3 = P_3 - \frac{(P_4 - P_3)^2}{P_5 - 2P_4 + P_3} = \underline{0.2575}911$$

$$\hat{P}_4 = P_4 - \frac{(P_5 - P_4)^2}{P_6 - 2P_5 + P_4} = \underline{0.2575}302$$

if 4 digit or more is similer So the Solving is Correct

c. $p_0 = 0.5, \quad p_n = 3^{-p_{n-1}}, \quad n \geq 1$

⇒ apply P_n formula :

$P_1 = 3^{-(P_0)} = 0.5773502$

$P_2 = 3^{-(P_1)} = 0.5303150$

$P_3 = 3^{-(P_2)} = 0.5584386$

$P_4 = 3^{-(P_3)} = 0.5414483$

$P_5 = 3^{-(P_4)} = 0.5516498$

$P_6 = 3^{-(P_5)} = 0.5455017$

⇒ apply P_6 in Aitken's formula:

$$\hat{P}_n = P_n - \frac{(P_{n+1} - P_n)^2}{P_{n+2} - 2P_{n+1} + P_n}$$

→ Start from $\hat{P}_0$:

$$\hat{P}_0 = P_0 - \frac{(P_1 - P_0)^2}{P_2 - 2P_1 + P_0} = 0.5481010$$

$$\hat{P}_1 = P_1 - \frac{(P_2 - P_1)^2}{P_3 - 2P_2 + P_1} = 0.5479150$$

$$\hat{P}_2 = P_2 - \frac{(P_3 - P_2)^2}{P_4 - 2P_3 + P_2} = \underline{0.5478}469$$

$$\hat{P}_3 = P_3 - \frac{(P_4 - P_3)^2}{P_5 - 2P_4 + P_3} = \underline{0.5478}225$$

$$P_4 = P_4 - \frac{(P_5 - P_4)^2}{P_6 - 2P_5 + P_4} = \underline{0.5478}136$$

4 digit are similer thus mean is correct

……

② Steffen's Method:

P_0^{o} and P_1^{o} mostly will be given in Q

→ to calculate P_2^{o} use $g(P_1^{o})$

when we use P in g(x)

P_1^{o}
o → does not change
1 → will change very time

if "P_n" is given in Q that means in g(x) form

but, if "P_n" Not given we need to convert f(x) to g(x)

→ apply formula

In formula P_0^1 will change

not change

$$P_0^1 = P_0^0 - \frac{(P_1^0 - P_0^0)^2}{P_2^0 - 2P_1^0 + P_0^0}$$

$P_1^1 \Rightarrow g(P_0^1)$ and $P_2^1 \Rightarrow g(P_1^1)$

$$P_0^2 = P_0^1 - \frac{(P_1^1 - P_0^1)^2}{P_2^1 - 2P_1^1 + P_0^1}$$

Q4a **4.** Let $g(x) = 1 + (\sin x)^2$ and $p_0^{(0)} = 1$. Use Steffensen's method to find $p_0^{(1)}$ and $p_0^{(2)}$.

→ find P_1^0 and P_2^0 :

$P_1^0 = g(P_0^0) = 1 + (\sin(1))^2 = 1.7080734$

$P_2^0 = g(P_1^0) = 1 + (\sin(P_1^0)^2) = 1.9812730$

$P_3^0 = g(P_2^0) = 1 + (\sin(P_2^0)^2) = 1.8407619$

→ find P_0^1 and P_0^2 by using formula:

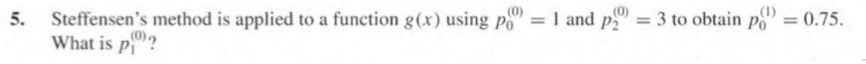
$$P_0^1 = P_0^0 - \frac{(P_1^0 - P_0^0)^2}{P_2^0 - 2P_1^0 + P_0^0} = \boxed{2.1529044}$$

$P_1^1 \Rightarrow g(P_0^1) = 1 + (\sin(P_0^1)^2 = 1.6977353$

$P_2^1 \Rightarrow g(P_1^1) = 1 + (\sin(P_1^1))^2 = 1.9839728$

$$P_0^2 = P_0^1 - \frac{(P_1^1 - P_0^1)^2}{P_2^1 - 2P_1^1 + P_0^1} = \boxed{1.8734640}$$

Q5:



5. Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1$ and $p_2^{(0)} = 3$ to obtain $p_0^{(1)} = 0.75$. What is $p_1^{(0)}$?

$$p_0^1 = p_0^0 - \frac{(p_1^0 - p_0^0)^2}{p_2^0 - 2p_1^0 + p_0^0}$$

$$\underline{0.75} = 1 \boxed{- \frac{(p_1^0 - 1)^2}{\underline{3} - 2p_1^0 + \underline{1}}}$$

$$\frac{(p_1^0 - 1)^2}{4 - 2p_1^0} = 1 - 0.75$$

$$\frac{(p_1^0 - 1)^2}{4 - 2p_1^0} = \frac{0.25}{1}$$

$$(p_1^0 - 1)^2 = (4 - 2p_1^0)(0.25)$$

$$(p_1^0 - 1)^2 = 1 - 0.5p_1^0$$

$$(p_1^0)^2 - 2p_1^0 + \cancel{1} = \cancel{1} - 0.5p_1^0$$

$$(p_1^0)^2 - 2p_1^0 = \underline{-0.5p_1^0}$$

$$(p_1^0)^2 - 2p_1^0 + 0.5p_1^0 = 0$$

$$p_1^0(p_1^0 - 2 + 0.5) = 0$$

$$p_1^0(p_1^0 - 1.5) = 0$$

$$\boxed{p_1^0 = 0} \qquad p_1^0 - 1.5 = 0$$

$$\boxed{p_1^0 = 1.5}$$

Q 78

7. Use Steffensen's method to find, to an $\boxed{\text{accuracy of } 10^{-4}}$, the root of $\underline{x^3 - x - 1 = 0}$ $\xrightarrow{} f(x)$ that lies in $[\overset{a}{1}, \overset{b}{2}]$ and compare this to the results of Exercise 8 of Section 2.2.

$\rightarrow$ $$f(x) = x^3 - x - 1$$

$$x^3 - x - 1 = 0$$ "from 2.2 fixed point"

$$(x^3)^{1/3} = (x + 1)^{1/3}$$

$$x = (x+1)^{1/3}$$

$$\downarrow$$

$$g(x) = (x+1)^{1/3}$$

$$g'(x) = \frac{1}{3}(x+1)^{-2/3} \cdot (1)$$

$$g'(x) = \frac{1}{3}(x+1)^{-2/3}$$

form Interval: $g'(a) = g'(1) = 0.2099868 \angle 1$

$$g'(b) = g(b) = 0.1602499 \angle 1$$

$$g(x) = (x+1)^{1/3} \Rightarrow P_n = (P_{n+1} + 1)^{1/3}$$

to find $p_0^0 \Rightarrow \frac{1+2}{2} = \boxed{1.5}$

$$p_1^0 = g(p_1^0) = (p_0^0 + 1)^{1/3} = 1.3572088$$

$$p_2^0 = g(p_1^0) = (p_1^0 + 1)^{1/3} = 1.3308609$$

find p_0^1 and p_0^2 :

$$p_0^1 = p_0^0 - \frac{(p_1^0 - p_0^0)^2}{p_2^0 - 2p_1^0 + p_0^0} = \boxed{1.3248990}$$

$$p_1^1 \Rightarrow g(p_0^1) = (p_0^1 + 1)^{1/3} = 1.3247523$$

$$p_2^1 \Rightarrow g(p_1^1) = (p_1^1 + 1)^{1/3} = 1.3247244$$

$$p_0^2 = p_0^1 - \frac{(p_1^1 - p_0^1)^2}{p_2^1 - 2p_1^1 + p_0^1} = \boxed{1.3287939}$$

$p_1^2 \Rightarrow g(p_0^2) = 1.3254917$

$p_2^2 \Rightarrow g(p_1^2) = 1.3248649$

$$p_0^3 = p_0^2 - \frac{(p_1^2 - p_0^2)^2}{p_2^2 - 2p_1^2 + p_0^2} = 1.3244180$$

$p_1^3 \Rightarrow g(p_0^3) = 1.3246609$

$p_2^3 \Rightarrow g(p_1^3) = 1.3247071$

$$p_0^4 = p_0^3 - \frac{(p_1^3 - p_0^3)^2}{p_2^3 - 2p_1^3 + p_0^3} = 1.3247179$$