Error Analysis for Iteration Methods

① Order of Convergence :

formula = $\boxed{\lim_{n \to \infty} \dfrac{|P_{n+1} - P|}{|P_n - P|^{\alpha}} = \lambda}$

linerly $\to \alpha = 1$

quadratically $\to \alpha = 2$

qupically $\to \alpha = 3$

when (α) is bigger number that's mean your approch to $(P) \to$ Root

Q6 :

6. Show that the following sequences converge linearly to $p = 0$. How large must n be before $|p_n - p| \leq 5 \times 10^{-2}$?

a. $p_n = \dfrac{1}{n}, \quad n \geq 1$

$$= \lim_{n \to \infty} \frac{|P_{n+1} - P|}{|P_n - P|^{\alpha}} = \lambda$$

$\Rightarrow$ If we have P_n we Can find P_{n+1} : $P_n = \dfrac{1}{n} \Rightarrow P_{n+1} = \dfrac{1}{n+1}$

from given in Q $\Rightarrow P = 0$, α = linearly = 1

to Solve limite Ignore (λ)

$$= \lim_{n \to \infty} \frac{\left|\frac{1}{n+1} - 0\right|}{\left|\frac{1}{n} - 0\right|^{1}}$$

$$= \lim_{n\to\infty} \frac{\left|\frac{1}{n+1}\right|}{\left|\frac{1}{n}\right|} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$ " to Solve limit use L'hopital rule "

by deividing the highest degree of (n)

$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}$$

Remark: $\rightarrow \lim_{n\to\infty} \frac{n}{n} = 1$

$\rightarrow \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$

$$= \lim \frac{1}{1+0} = \boxed{1}$$

So P_n is Convergences linearly in $p=0$

b. $p_n = \frac{1}{n^2}, \quad n \geq 1$

$\Rightarrow$ If we have P_n we Can find P_{n+1}: $P_n = \frac{1}{n^2} \Rightarrow P_{n+1} = \frac{1}{n^2+1}$

$$\lim_{n\to\infty} \frac{\left|\frac{1}{n^2+1} - 0\right|}{\left|\frac{1}{n^2} - 0\right|} \Rightarrow \lim_{n\to\infty} \frac{\left|\frac{1}{n^2+1}\right|}{\left|\frac{1}{n^2}\right|} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{1}{n^2+1} \cdot \frac{n^2}{1} \Rightarrow \lim_{n\to\infty} \frac{n^2}{n^2+1}$$ "use L'hoptical rule"

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}} \Rightarrow \lim_{n\to\infty} \frac{1}{1+0} = \boxed{1}$$

So P_n is Convergences linearly in $p=0$

Q8: **8.** **a.** Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to 0.

$\alpha = 2$ $\quad$ $P = 0$

$$\lim_{n\to\infty} \frac{|P_{n+1} - P|}{|P_n - P|^{\alpha}} = \lambda$$

$\Rightarrow$ If we have P_n we can find P_{n+1} : $\quad P_n = 10^{-2^n} \Rightarrow P_{n+1} = 10^{-2^{n+1}}$

$P_{n+1} = 10^{-2n+1} = 10^{-2n} \times 10^{n}$

given in Q $\Rightarrow$ $\alpha = 2$ and $P = 0$

$$\lim_{n\to\infty} \frac{|10^{-2^{n+1}} - 0|}{|10^{-2^n} - 0|^2}$$

$$= \lim_{n\to\infty} \frac{|10^{-2^{n+1}}|}{|10^{-2^n}|^2}$$

$$= \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{|10^{-2^n}|^2} \Rightarrow \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^n \cdot 2^1}}$$

" If they've same base we can add powers $2^n \cdot 2^1 = 2^{n+1}$ "

$$= \lim_{n\to\infty} \frac{\cancel{10^{-2^{n+1}}}}{\cancel{10^{-2^{n+1}}}}$$

$$\lim_{n\to\infty} 1 = \boxed{1}$$

So P_n is Convergence in $P=0$

Q9: **9.** **a.** Construct a sequence that converges to 0 of order 3.

$P_n = 7^{-3^n}$

Base Can be any number

Should match order α w/ Power of P_n

$\Rightarrow$ If we have P_n we can find P_{n+1} : $P_n = 7^{-3^n}$ $\Rightarrow$ $P_{n+1} = 7^{-3^{n+1}}$

$$\Rightarrow \lim_{n\to\infty} \frac{|7^{-3^{n+1}} - 0|}{|7^{-3^n} - 0|^3} \Rightarrow \lim_{n\to\infty} \frac{|7^{-3^{n+1}}|}{|7^{-3^n}|^{\boxed{3}}} \Rightarrow \lim_{n\to\infty} \frac{7^{-3^{n+1}}}{7^{-3^n \cdot 3}}$$

$$\lim_{n\to\infty} \frac{\cancel{7^{-3^{n+1}}}}{\cancel{7^{-3^{n+1}}}} = 1$$

$$\lim_{n\to\infty} 1 = \boxed{1}$$

so P_n is Convergence in $p=0$

Q3 :

3. Repeat Exercise 1 using the modified Newton's method described in Eq. (2.13). Is there an improvement in speed or accuracy over Exercise 1?

a. $x^2 - 2xe^{-x} + e^{-2x} = 0, \quad \text{for } 0 \le x \le 1$

modified Newton's method $\Rightarrow$ $P_n = P_{n-1} - \dfrac{f(P_{n-1})\, f'(P_{n-1})}{f'(P_{n-1})^2 - f(P_{n-1})\, f''(P_{n-1})}$

$\Rightarrow$ $f(x) = x^2 - 2e^{-x} + e^{-2x}$, $[0,1]$