Algorithms and Convergence

① Rate of Convergence for Sequences :

Formula : $|\alpha_n - \alpha| \le K|\beta_n|$, for large n, , $\beta_n = \dfrac{1}{n^p}$,

6. Find the rates of convergence of the following sequences as $n \to \infty$.

a. $\lim_{n\to\infty} \sin \dfrac{1}{n} = 0$ x

to solve Q use ⇒

$\boxed{|\sin x| \leqslant |x|, \text{ where } 0 \leqslant x < 1}$

$= \left|\sin \frac{1}{n}\right| \underset{K=1}{\leqslant} \left|\frac{1}{n}\right|$ $\quad \beta_n = \frac{1}{n}$

by using formula

$$\therefore \lim_{n\to\infty} \sin \frac{1}{n} = O\left(\frac{1}{n}\right)$$

b. $\lim_{n\to\infty} \sin \dfrac{1}{n^2} = 0$

$\left|\sin \frac{1}{n^2}\right| \underset{K=1}{\leqslant} \left|\frac{1}{n^2}\right|$ $\quad \beta_n = \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty} \sin \frac{1}{n^2} = O\left(\frac{1}{n^2}\right)$$

c. $\lim_{n\to\infty} \left(\sin\frac{1}{n}\right)^2 = 0$

note: Sequar and Absolut value Can Cansel each other

$$= \left|\left(\sin\frac{1}{n}\right)^2\right| \underset{K=1}{\leq} \left|\left(\frac{1}{n}\right)^2\right| = \underbrace{\left(\frac{1}{n^2}\right)}_{B_n}$$

$$\therefore \lim_{n\to\infty} \left(\sin\frac{1}{n}\right)^2 = O\left(\frac{1}{n^2}\right)$$

d. $\lim_{n\to\infty} [\ln(n+1) - \ln(n)] = 0$

to solve Q use ⇒

$|\ln(1+x)| \leq |x|$, where $0 < x < 1$

$$= [\ln(n+1) - \ln(n)] = \ln\frac{n+1}{n} = \ln\left(\frac{n}{n} + \frac{1}{n}\right) = \ln\left(1 + \frac{1}{n}\right)$$

$$\Rightarrow \left|\ln\left(1+\frac{1}{n}\right)\right| \underset{K=1}{\leq} \left|\underbrace{\left(\frac{1}{n}\right)}_{B_n}\right|$$

$\ln(ab) = \ln(a) + \ln(b)$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$\ln(a^b) = b\ln(a)$

$$\therefore \lim_{n\to\infty} [\ln(n+1) - \ln(n)] = O\left(\frac{1}{n}\right)$$

② Rate of Convergence for function :

formula : $|F(h) - L| \leq K|G(h)|$ ⇒ $F(h) = L + O(G(h))$. where $G(h) = h^p$,

Maclurin Series : $\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!} + \dots$

$$\frac{}{h} \quad \frac{\qquad}{h}$$

need the first two Values

$$\boxed{\frac{\cos h}{h} = \frac{1 - \frac{h^2}{2} + \frac{h^3}{3} + \dots}{h}}$$

$$\boxed{\frac{e^h}{h} = \frac{1 + \frac{h^2}{2} + \frac{h^3}{3} + \dots}{h}}$$

7. Find the rates of convergence of the following functions as $h \to 0$.

a. $\lim_{h\to 0} \frac{\sin h}{h} = 1 \to L$

$$= \frac{\sin h}{h} = \frac{h - \frac{h^3}{6} + \frac{h^5}{120}}{h} = \frac{h}{h} - \frac{\frac{h^3}{6}}{h} = 1 - \boxed{\frac{h^2}{6}}$$ use the first (h)

$$\text{so } \lim_{h\to 0} \frac{\sin h}{h} = L + O(h^p) = 1 + O(h^2)$$

b. $\lim_{h\to 0} \frac{1-\cos h}{h} = 0 \to L$ $\quad \boxed{\frac{\cos h}{h} = \frac{1 - \frac{h^2}{2} + \frac{h^3}{3} + \dots}{h}}$

$$\frac{1-\cos h}{h} = \frac{1 - (1 - \frac{h^2}{2} + \frac{h^3}{3} + \dots)}{h} = \frac{\cancel{1} \cancel{-1} + \frac{h^2}{2} - \frac{h^3}{3} + \dots}{h} = \frac{\frac{h^2}{2}}{h} - \frac{\frac{h^3}{3}}{h}$$

$$\frac{1-\cos h}{h} = \boxed{\frac{h}{2}} - \frac{h^2}{3}$$

use the first (h)

$$\therefore \lim_{h \to 0} \frac{1-\cos h}{h} = 0 + O(h) = O(h)$$

c. $\lim_{h \to 0} \frac{\sin h - h\cos h}{h} = 0 \rightarrow L$

$$\frac{\sin h - h\cos h}{h} = \frac{\left(h - \frac{h^3}{6} + \cdots\right) - h\left(1 - \frac{h^2}{2} + \cdots\right)}{h}$$

$$= \frac{\left(\cancel{h} - \frac{h^3}{6}\right)\left(-\cancel{h} + \frac{h^3}{2}\right)}{h} = \frac{-h^3}{6} + \frac{h^3 \times 3}{2 \times 3} = \frac{-h^3 + 3h^3}{6} = \boxed{\frac{2h^3}{6}}$$

take (h^3)

$$\therefore \lim_{h \to 0} \frac{\sin h - h\cos h}{h} = 0 + O(h^3) = O(h^3)$$

d. $\lim_{h \to 0} \frac{1 - e^h}{h} = -1 \rightarrow L$

$$\frac{e^h}{h} = \frac{1 + \frac{h^2}{2} + \frac{h^3}{3} + \cdots}{h}$$

$$= \frac{1 - e^h}{h} = \frac{1 - \left(1 + \frac{h^2}{2}\right)}{h} = \frac{\cancel{1} \cancel{-1} - \frac{h^2}{2}}{h} = \frac{-\frac{h^2}{2}}{h} = \boxed{\frac{-h}{2}} \rightarrow \text{take } (h) \text{ w/o sign}$$

$$\therefore \lim_{h \to 0} \frac{1 - e^h}{h} = -1 + O(h)$$