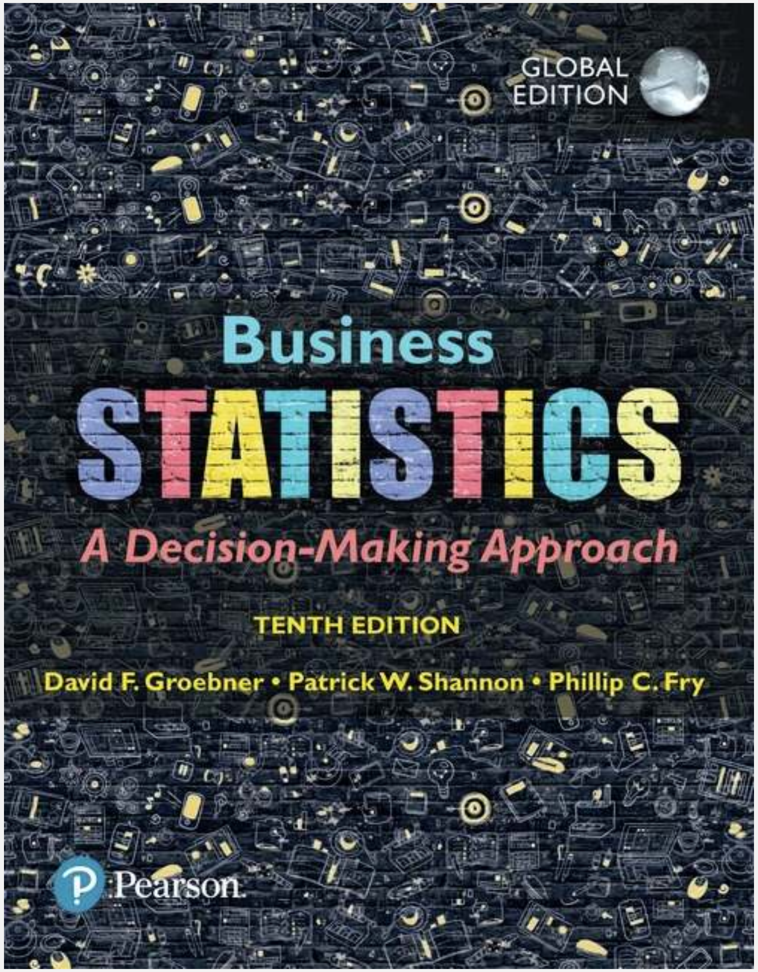
Chapter 5

Discrete Probability Distributions



Learning Outcomes

Outcome 1. Be able to calculate and interpret the expected value of a discrete random variable.

Outcome 2. Be able to apply the binomial distribution to business decision-making situations.

Outcome 3. Be able to compute probabilities for the Poisson and hypergeometric distributions and apply these distributions to decision-making situations.

5.1 Introduction to Discrete Probability Distributions

- Random Variable
  - Takes on different numerical values based on chance
- Discrete Random Variable
  - Can only assume a finite number of values or an infinite sequence of values such as 0,1,2,…
- Continuous Random Variable
  - Can assume an uncountable infinite number of values

Discrete Random Variable

- **Many possible outcomes:**
  - number of complaints per day
  - number of TVs in a household
  - number of rings before the phone is answered
- **Only two possible outcomes:**
  - gender: male or female
  - defective item: yes or no
  - game result: won or lost

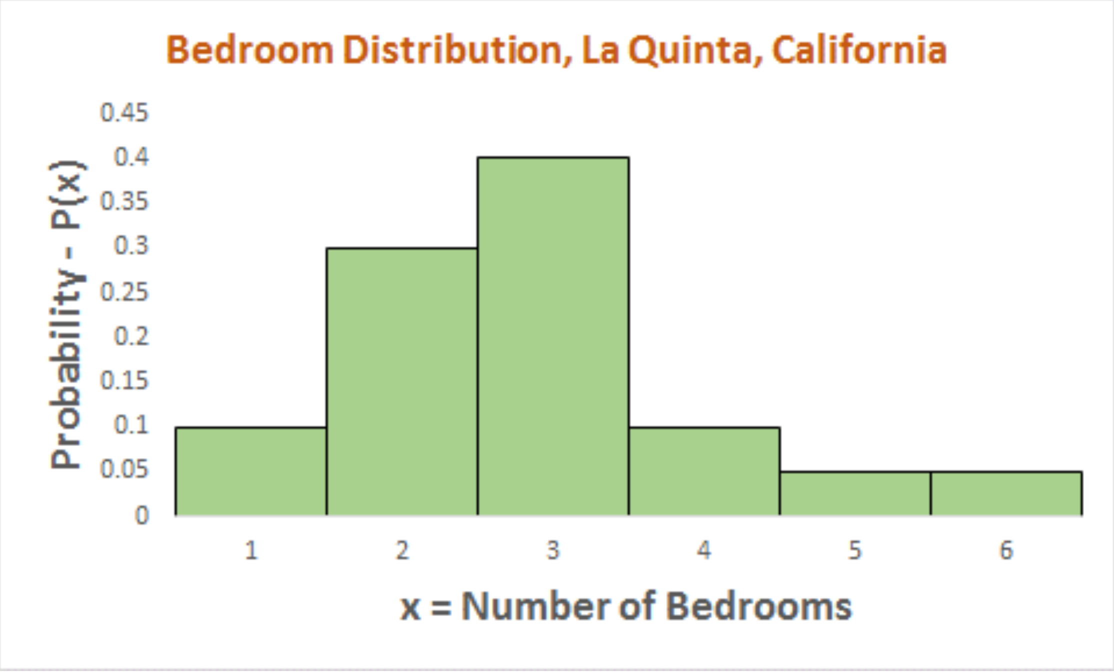
Discrete Random Variable - Example

Real Estate Listings in La Quinta, California show homes with various numbers of bedrooms. The variable of interest is x = number of bedrooms. The probability of each possible outcome is based on relative frequency of occurrence.

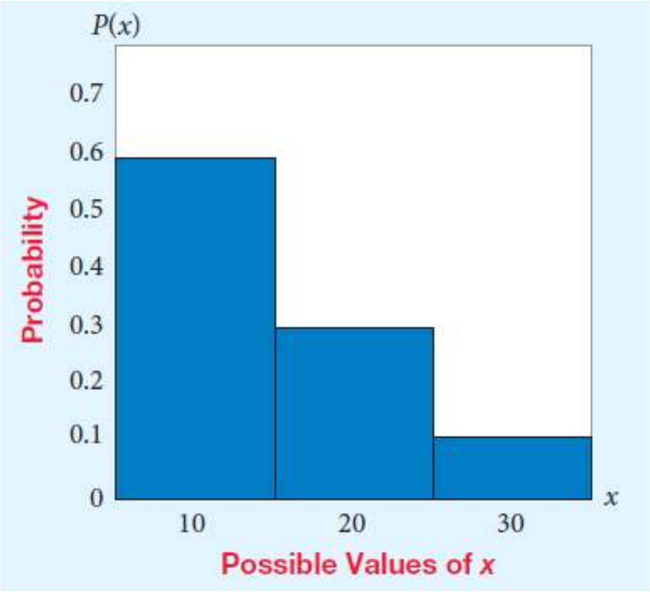
x =number of bedrooms	P(x)
1	0.10
2	0.30
3	0.40
4	0.10
5	0.05
6	0.05
Sum =	1.00

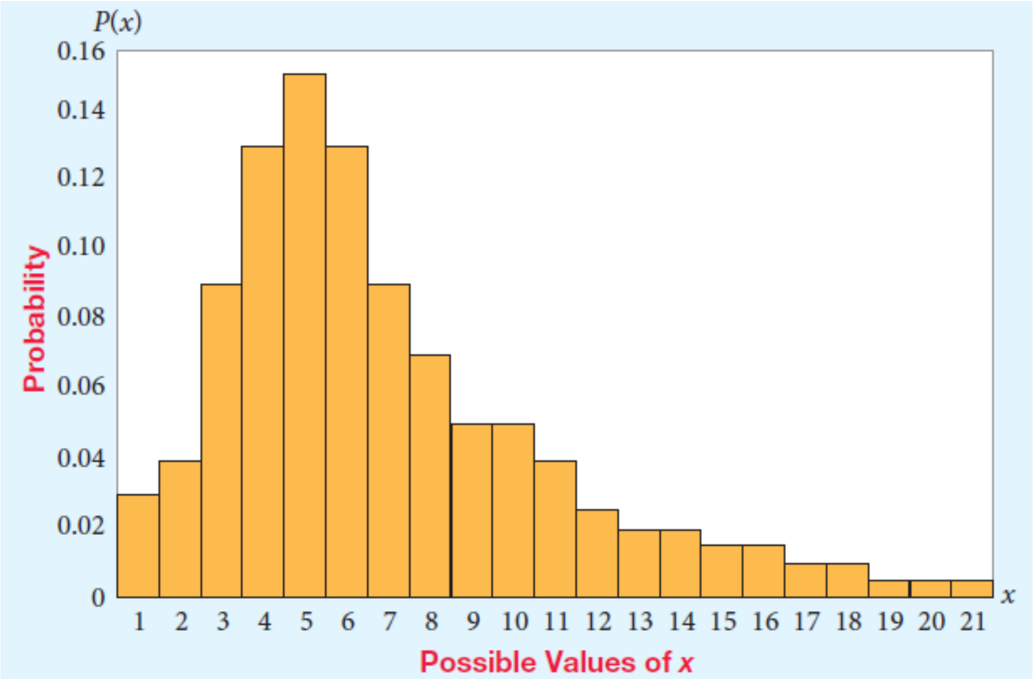
Discrete Random Variable – Example
Graphical Presentation of Probability Distribution

x =number of bedrooms	P(x)
1	0.10
2	0.30
3	0.40
4	0.10
5	0.05
6	0.05
Sum =	1.00



Discrete Probability Distributions



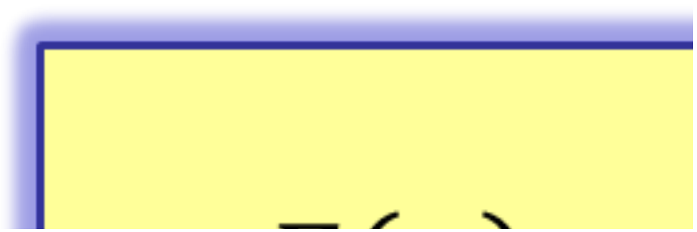


3 possible outcomes

21 possible outcomes

Expected Value of a Discrete Probability Distribution

- The mean of a discrete probability distribution. The average value when the experiment that generates values for the discrete random variable is repeated over the long run.



$E(x)$ = Expected value of x
x = Values of the discrete random variable
$P(x)$ = Probability of the random variable taking on the value x

Expected Value of a Discrete Random Variable - Example

Number of Bedrooms in Home for Sale – Calculate the Expected Value.

x =number of bedrooms	P(x)	xP(x)
1	0.10	0.10
2	0.30	0.60
3	0.40	1.20
4	0.10	0.40
5	0.05	0.25
6	0.05	0.30
	E(x) =	2.85

The expected number of bedrooms is 2.85 which represents the mean of the probability distribution.

Expected Value of a Discrete Random Variable - Example

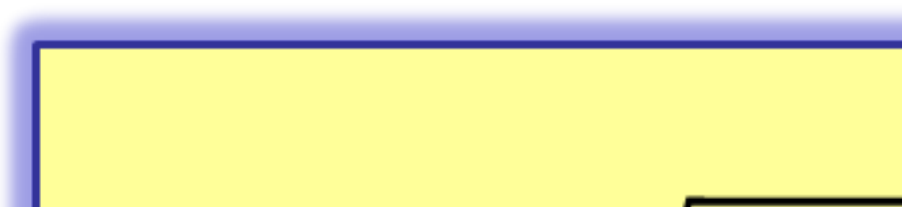
Suppose you are given a chance to pay $3 to play a game in which one 6-sided die is rolled and you receive a payout equal to $1 times the value that comes up on the die. What is the expected profit for this game?

You could expect to win $.50 on average each time you play the game.

x =payoff	P(x)	xP(x)
($2)	0.167	$(0.33)
($1)	0.167	$(0.17)
$0	0.167	$ -
$1	0.167	$ 0.17
$2	0.167	$ 0.33
$3	0.167	$ 0.50
	E(x) =	$ 0.50

Standard Deviation of a Discrete Random Variable

- Standard deviation measures the spread, or dispersion, in a set of data.
- The standard deviation also measures the spread in the values of a random variable.



$E(x)$ = Expected value of x
x = Values of the discrete random variable
$P(x)$ = Probability of the random variable taking on the value x

Standard Deviation of a Discrete Random Variable - Example

Number of Bedrooms in Home for Sale – Calculate the Standard Deviation

x =number of bedrooms	P(x)	[x-E(x)]	$[x-E(x)]^2$	$[x-E(x)]^2P(x)$
1	0.10	-1.85	3.4225	0.342
2	0.30	-0.85	0.7225	0.217
3	0.40	0.15	0.0225	0.009
4	0.10	1.15	1.3225	0.132
5	0.05	2.15	4.6225	0.231
6	0.05	3.15	9.9225	0.496
			sum =	1.427

Standard Deviation of a Discrete Random Variable - Example

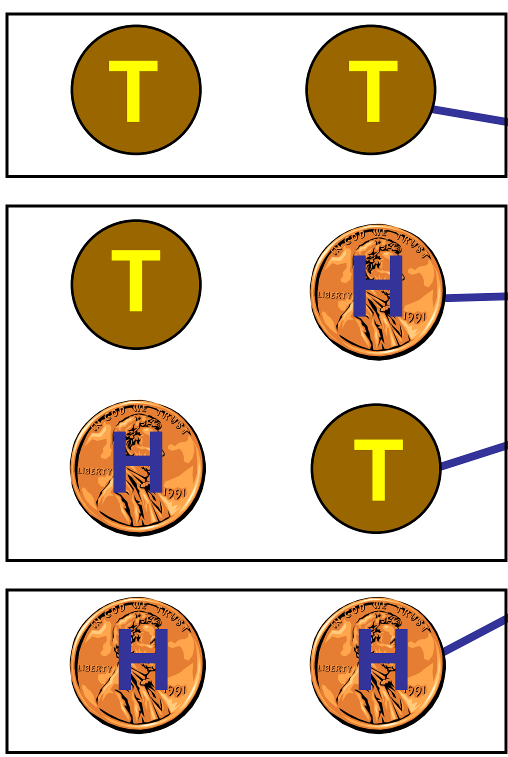
Dice Game – Pay $3 to play – win $1 times the value on the die

x =payoff	P(x)	[x-E(x)]	$[x\text{-}E(x)]^2$	$[x\text{-}E(x)]^2P(x)$
($2)	0.1667	($2.50)	6.25	1.044
($1)	0.167	($1.50)	2.25	0.376
$0	0.167	($0.50)	0.25	0.042
$1	0.167	$0.50	0.25	0.042
$2	0.167	$1.50	2.25	0.376
$3	0.167	$2.50	6.25	1.044
			sum =	2.923

Expected Value and Standard Deviation Example

Experiment: Toss 2 Coins. x = Number of heads

4 possible outcomes



x-value	Probability
0	1/4 = 0.25
1	2/4 = 0.50
2	1/4 = 0.25

Expected Value and Standard Deviation Example

Experiment: Toss 2 Coins. x = Number of heads
Compute expected value and standard deviation

Expected Value

$E(x) = (0 \times 0.25) + (1 \times 0.50) + (2 \times 0.25) = 1.0$

x	$P(x)$
0	0.25
1	0.50
2	0.25

Standard Deviation

Expected Value and Standard Deviation Example

Calculate the Expected Payoff and Standard Deviation for Each Investment Option.

Investment 1		Investment 2		Investment 3	
[illegible]f	Probability	Payoff	Probability	Payoff	Probability
$ 20	0.25	$ 20	0.2	$ 5	0.2
$ 30	0.25	$ 30	0.3	$ 10	0.3
[illegible]0	0.25	$ 40	0.3	$ 30	0.4
[illegible]0	0.25	$ 50	0.2	$ 100	0.1

Do you prefer 1 or 2? Why?

Do you prefer 1, 2, or 3? Why?

Expected Value and Standard Deviation Example

Investment # 1

x= Payoff	P(x)	xP(x)	[x-E(x)]	$[x-E(x)]^2$	$[x-E(x)]^2P(x)$
$20	0.25	$5.00	($15)	225	56.25
$30	0.25	$7.50	($5)	25	6.25
$40	0.25	$10.00	$5	25	6.25
$50	0.25	$12.50	$15	225	56.25
	E(x) =	$35		Sum =	125.0

Expected Value and Standard Deviation Example

Investment # 2

x= Payoff	P(x)	xP(x)	[x-E(x)]	$[x-E(x)]^2$	$[x-E(x)]^2P(x)$
$20	0.2	$4.00	($15)	225	45.00
$30	0.3	$9.00	($5)	25	7.50
$40	0.3	$12.00	$5	25	7.50
$50	0.2	$10.00	$15	225	45.00
	E(x) =	$35		Sum =	105.0

Expected Value and Standard Deviation Example

Investment # 3

x= Payoff	P(x)	xP(x)	[x-E(x)]	$[x-E(x)]^2$	$[x-E(x)]^2P(x)$
$5	0.2	$1	($21)	441	88.2
$10	0.3	$3	($16)	256	76.8
$30	0.4	$12	$4	16	6.4
$100	0.1	$10	$74	5476	547.6
	E(x) =	$26		Sum =	719.0

5.2 The Binomial Probability Distribution

- A distribution that gives the probability of *x* successes in *n* trials in a process that meets the following conditions:

1. A trial has only **two** possible outcomes: a success or a failure.
2. The [illegible] a **fixed** number, *n*, of identical trials.
3. The trials of the experiment are **independent** of each other. This means that if one outcome is a success, this does not influence the chance of another outcome being a success.
4. The process must be **consistent** in generating successes and failures. That is, the probability, *p*, associated with a success remains constant from trial to trial.
5. If *p* represents the probability of a success, then $(1 – p) = q$ is the probability of a failure.

Binomial Distribution Examples

- A manufacturing company inspects products and classifies them as either defective or acceptable
- A firm bidding for a contract will either get the contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it

Binomial Distribution Example

A Bank believes that 80% of its customers prefer the bank to close early on Friday but be open on Saturday morning. Suppose three customers are randomly surveyed and that the binomial distribution applies. Develop the probability distribution for the number of these customers who like the Saturday option.

Events: S =prefer Saturday F = prefer longer Friday

x = Number who prefer Saturday
0
1
2
3

Binomial Distribution - Bank Example

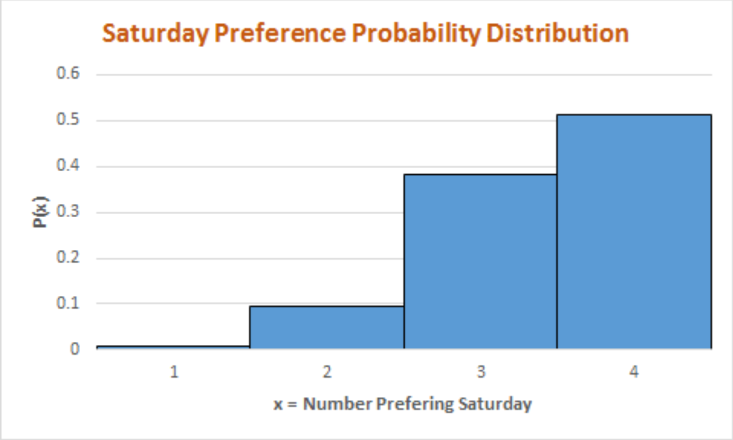
Begin by listing the Sample Space

x = number of S customers

Binomial Distribution - Bank Example

Assign Probabilities Using Multiplication Rule for Independent Events and the Addition Rule

x = Number who prefer Saturday	P(x)
0	0.008
1	0.096
2	0.384
3	0.512
	1.000



Binomial Distribution - Bank Example

Suppose the bank interviews 8 customers rather than 3. It's too hard to list out the sample space. We need an alternative method for determining the binomial distribution.

x = Number who prefer Saturday
0
1
2
3
4
5
6
7
8

Counting Rule for Combinations

- A method for counting the number of ways binomial events can occur

$n! = n(n - 1)(n - 2) \ldots (2)(1)$ $\quad$ $x! = x(x - 1)(x - 2) \ldots (2)(1)$

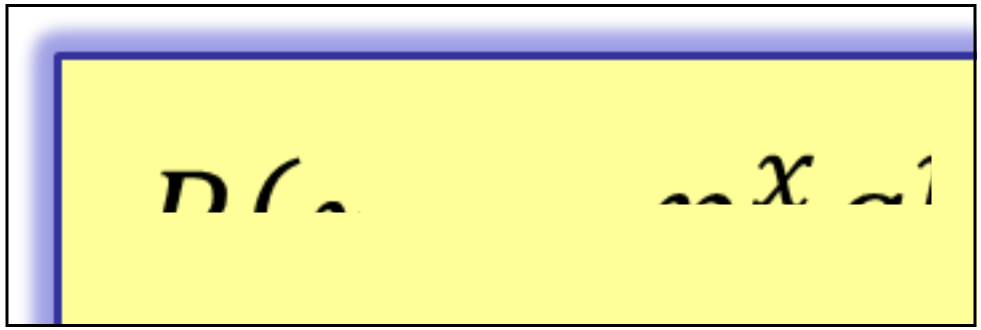
$0! = 1$ by definition

The order of events does not matter.

Binomial Distribution - Bank Example
Interview n=8 customers

x = Number who prefer Saturday	# Ways the Event Can Occur
0	1
1	8
2	28
3	56
4	70
5	56
6	28
7	8
8	1

Binomial Formula



$P(x)$ - probability of x successes in n trials,
with probability of success p on each trial

x - number of *successes* in sample,
(x = 0, 1, 2, ..., n)

p - probability of a success

q - probability of a failure $q = 1 - p$

n - number of trials (sample size)

Binomial Distribution - Bank Example Interview n=8 customers

Use Binomial Formula t
x = 3 customers in a sa
Saturday opening.

$P(S) = p = 0.80$

Binomial Distribution - Bank Example
Interview n=8 customers

x = Number who prefer Saturday	P(x)
0	0.000
1	0.000
2	0.001
3	0.009
4	0.046
5	0.147
6	0.294
7	0.336
8	0.168
	1.000

$$P(x) = \frac{n!}{x!(n-x)!} p^x$$

Using the Cumulative Binomial Table

35% of the population support a new regulation, if a random sample of 10 voters is polled, what is the probability that **three or fewer** of them support the proposition?

					n = 10			
x	p=.15	p=.20	p=.25	p=.30	**p=.35**	p=.40	p=.45	p=.50
0	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010
1	0.5443	0.3758	0.2440	0.1493	0.0860	0.0464	0.0233	0.0107
2	0.8202	0.6778	0.5256	0.3828	0.2616	0.1673	0.0996	0.0547
3	0.9500	0.8791	0.7759	0.6496	**0.5138**	0.3823	0.2660	0.1719
4	0.9901	0.9672	0.9219	0.8497	0.7515	0.6331	0.5044	0.3770
5	0.9986	0.9936	0.9803	0.9527	0.9051	0.8338	0.7384	0.6230
6	0.9999	0.9991	0.9965	0.9894	0.9740	0.9452	0.8980	0.8281
7	1.0000	0.9999	0.9996	0.9984	0.9952	0.9877	0.9726	0.9453
8	1.0000	1.0000	1.0000	0.9999	0.9995	0.9983	0.9955	0.9893
9	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

$n = 10, p = 0.35, x \leq 3$:

$P(x \leq 3, n = 10, p = 0.35) = 0.5138$

Finding an Exact Probability Using the Cumulative Binomial Table

An airline has determined that the probability that a customer who has purchased a first-class ticket between San Francisco and Boston will not show for the flight is 0.08. Suppose the airline has sold 10 first-class tickets for an upcoming flight. What is the probability that exactly 2 people will be no-shows?

Finding an Exact Probability Using the Cumulative Binomial Table

$n = 10$									
x	$p = 0.01$	$p = 0.02$	$p = 0.03$	$p = 0.04$	$p = 0.05$	$p = 0.06$	$p = 0.07$	$p = 0.08$	$p = 0.09$
0	0.9044	0.8171	0.7374	0.6648	0.5987	0.5386	0.4840	0.4344	0.3894
1	0.9957	0.9838	0.9655	0.9418	0.9139	0.8824	0.8483	0.8121	0.7746
2	0.9999	0.9991	0.9972	0.9938	0.9885	0.9812	0.9717	0.9599	0.9460
3	1.0000	1.0000	0.9999	0.9996	0.9990	0.9980	0.9964	0.9942	0.9912
4	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9994	0.9990
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

$n = 10$; $p = 0.08$

Mean and Standard Deviation for a Binomial Distribution

Mean and Standard Deviation for a Binomial Distribution

Bank Example

x = Number who prefer Saturday	P(x)	xP(x)	[x-E(x)]	$[x-E(x)]^2$	$[x-E(x)]^2P(x)$
0	0.000	0.00	(6.400)	40.960	0.000
1	0.000	0.00	(5.400)	29.160	0.000
2	0.001	0.00	(4.400)	19.360	0.019
3	0.009	0.03	(3.400)	11.560	0.104
4	0.046	0.18	(2.400)	5.760	0.265
5	0.147	0.74	(1.400)	1.960	0.288
6	0.294	1.76	(0.400)	0.160	0.047
7	0.336	2.35	0.600	0.360	0.121
8	0.168	1.34	1.600	2.560	0.430
	E(x) =	6.40		sum =	1.275

For
Binomial

Using the Binomial Distribution in Quality Testing

Situation:

Contract with Supplier Calls for the purchase of 3,000 components

Defect Rate: = 0.05 = Good Shipment

Defect Rate: = 0.10 = Bad Shipment

Acceptance Sampling Plan:

$n = 20$

If $x \leq 1$ defect, accept the shipment.

If $x > 1$ defect, reject the shipment.

Using the Binomial Distribution in Quality Testing

Objectives:

High Probability of Accepting Good Shipments

Low Probability of Accepting Bad Shipments

Using the Cumulative Binomial Table

				$n = 20$					
x	$p = 0.01$	$p = 0.02$	$p = 0.03$	$p = 0.04$	$p = 0.05$	$p = 0.06$	$p = 0.07$	$p = 0.08$	$p = 0.09$
0	0.8179	0.6676	0.5438	0.4420	0.3585	0.2901	0.2342	0.1887	0.1516
1	0.9831	0.9401	0.8802	0.8103	0.7358	0.6605	0.5869	0.5169	0.4516
2	0.9990	0.9929	0.9790	0.9561	0.9245	0.8850	0.8390	0.7879	0.7334
3	1.0000	0.9994	0.9973	0.9926	0.9841	0.9710	0.9529	0.9294	0.9007
4	1.0000	1.0000	0.9997	0.9990	0.9974	0.9944	0.9893	0.9817	0.9710
5	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9981	0.9962	0.9932
6	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9994	0.9987
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Assume the Shipment is Good (defect rate = 0.05)

$n = 20$ $p = 0.05$ Find $P(x \leq 1) = ?$

$P(x \leq 1) = 0.7358$ = Probability of accepting a Good shipment

Using the Cumulative Binomial Table

n= 20

x	p = 0.10	p = 0.15	p = 0.20	p = 0.25	p = 0.30	p = 0.35	p = 0.40	p = 0.45	p = 0.50
0	0.1216	0.0388	0.0115	0.0032	0.0008	0.0002	0.0000	0.0000	0.0000
1	0.3917	0.1756	0.0692	0.0243	0.0076	0.0021	0.0005	0.0001	0.0000
2	0.6769	0.4049	0.2061	0.0913	0.0355	0.0121	0.0036	0.0009	0.0002
3	0.8670	0.6477	0.4114	0.2252	0.1071	0.0444	0.0160	0.0049	0.0013
4	0.9568	0.8298	0.6296	0.4148	0.2375	0.1182	0.0510	0.0189	0.0059
5	0.9887	0.9327	0.8042	0.6172	0.4164	0.2454	0.1256	0.0553	0.0207
6	0.9976	0.9781	0.9133	0.7858	0.6080	0.4166	0.2500	0.1299	0.0577
7	0.9996	0.9941	0.9679	0.8982	0.7723	0.6010	0.4159	0.2520	0.1316
8	0.9999	0.9987	0.9900	0.9591	0.8867	0.7624	0.5956	0.4143	0.2517
9	1.0000	0.9998	0.9974	0.9861	0.9520	0.8782	0.7553	0.5914	0.4119
10	1.0000	1.0000	0.9994	0.9961	0.9829	0.9468	0.8725	0.7507	0.5881
11	1.0000	1.0000	0.9999	0.9991	0.9949	0.9804	0.9435	0.8692	0.7483
12	1.0000	1.0000	1.0000	0.9998	0.9987	0.9940	0.9790	0.9420	0.8684
13	1.0000	1.0000	1.0000	1.0000	0.9997	0.9985	0.9935	0.9786	0.9423
14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9936	0.9793

Assume the Shipment is Bad (defect rate = 0.10)

$n = 20 \quad p = 0.10$

$P(x \leq 1) = 0.3917$ = Probability of accepting a Bad shipment

Using the Binomial Distribution in Quality Testing

Objectives:

High Probability of Accepting Good Shipments

P(Accepting a Good Shipment) = 0.7358

Low Probability of Accepting Bad Shipments

P(Accepting a Bad Shipment) = 0.3917

What do you think of the Sampling Plan?

5.3 Other Discrete Probability Distributions

- Poisson Distribution
  - Describes a process that extends over space, time, or any well defined segment/interval or unit of inspection in which the outcomes of interest occur at random and the number of outcomes that occur in any given interval are counted
  - Is used when the total number of possible outcomes cannot be determined

Characteristics of the Poisson Distribution

- The outcomes of interest are rare relative to the possible outcomes.
- The average number of outcomes of interest per segment interval is .
- The number of outcomes of interest are random, and the occurrence of one outcome does not influence the chances of another outcome of interest.
- The probability that an outcome of interest occurs in a given segment is the same for all segments.

Poisson Distribution Examples

- Number of patients who arrive at a hospital emergency room per hour
- Number of calls the Dell technical support receives in a 30-minute period
- Number of cars served at the gas station in 24-hour period
- Number of defects per square meter of leather

The Poisson Distribution

- Mean or Expected Value

Standard Deviation

- Probability Distribution

t - Number of segments of interest

x - Number of successes in *t* segments

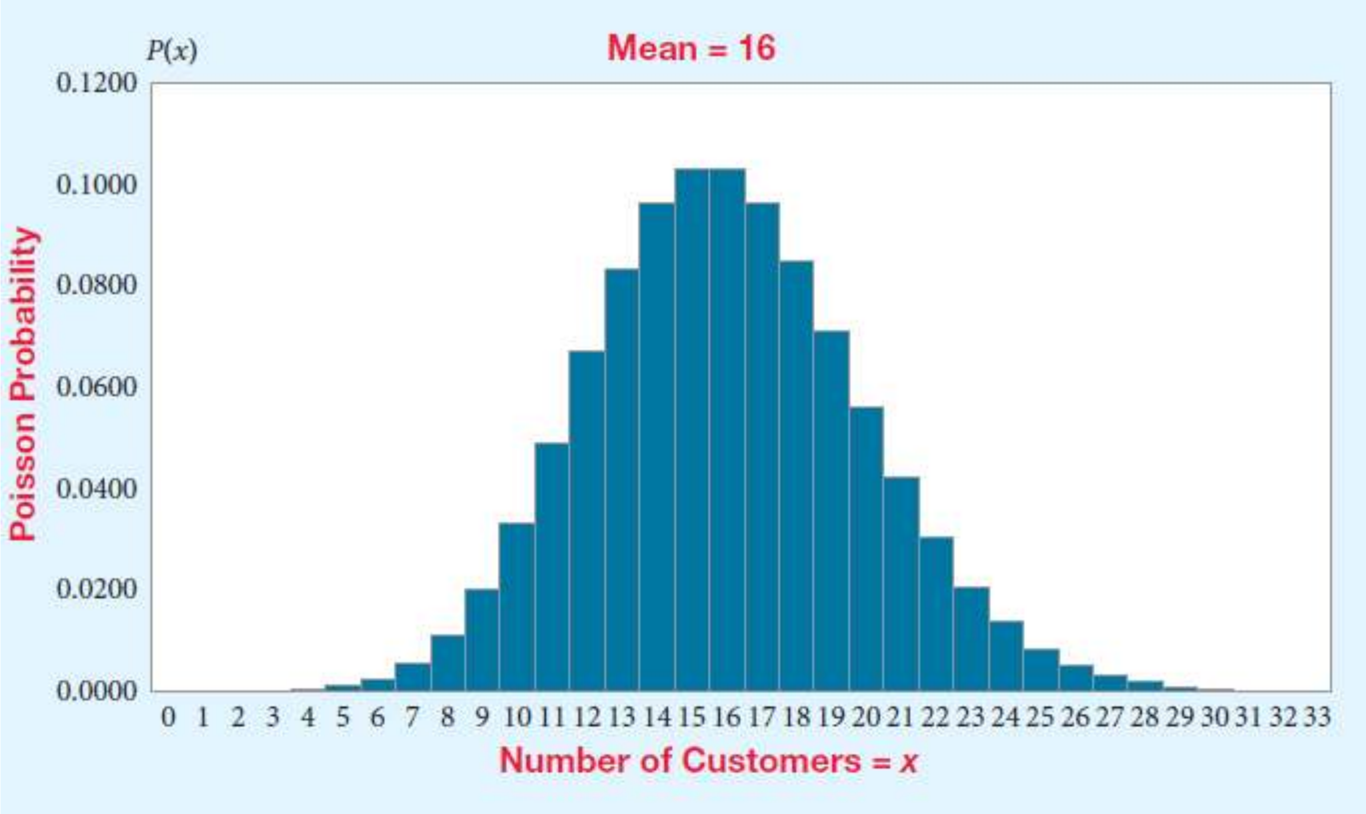
λ - Expected number of successes in one segment

e - Base of the natural logarithm system (2.71828)

Poisson Distribution

Distribution for the number of arrivals to the checkout section of the store per hour ($t=1$), λ = 16.

$$P(x = 12) = \frac{16^{12}e^{-16}}{12!} = 0.0661$$



Using the Poisson Distribution

- Step 1: Define the segment units. The segment units are usually blocks of time, areas of space, or volume.
- Step 2: Determine the mean of the random variable.
- Step 3: Determine t, the number of the segments to be considered, and then calculate λt.
- Step 4: Define the event of interest and use the Poisson formula or the Poisson table to find the probability.

Poisson Distribution - Example

At a Major League Baseball park, five customers arrive at a concession stand on average in a ten minute period. What is the probability that in a ten minute period <u>exactly three</u> people will arrive at the concession stand?

Segment size = 10 minutes

Poisson Distribution - Example

Baseball concession example – Find Probability of 3 or fewer customers arriving in 20 minutes?

Two Segments

For example,

Using the Cumulative Poisson Table

Baseball concession example – Find Probability of 3 or fewer customers arriving in 20 minutes?

	λt									
x	9.10	9.20	9.30	9.40	9.50	9.60	9.70	9.80	9.90	10.00
0	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000
1	0.0011	0.0010	0.0009	0.0009	0.0008	0.0007	0.0007	0.0006	0.0005	0.0005
2	0.0058	0.0053	0.0049	0.0045	0.0042	0.0038	0.0035	0.0033	0.0030	0.0028
3	0.0198	0.0184	0.0172	0.0160	0.0149	0.0138	0.0129	0.0120	0.0111	0.0103
4	0.0517	0.0486	0.0456	0.0429	0.0403	0.0378	0.0355	0.0333	0.0312	0.0293
5	0.1098	0.1041	0.0986	0.0935	0.0885	0.0838	0.0793	0.0750	0.0710	0.0671
6	0.1978	0.1892	0.1808	0.1727	0.1649	0.1574	0.1502	0.1433	0.1366	0.1301
7	0.3123	0.3010	0.2900	0.2792	0.2687	0.2584	0.2485	0.2388	0.2294	0.2202
8	0.4426	0.4296	0.4168	0.4042	0.3918	0.3796	0.3676	0.3558	0.3442	0.3328
9	0.5742	0.5611	0.5479	0.5349	0.5218	0.5089	0.4960	0.4832	0.4705	0.4579
10	0.6941	0.6820	0.6699	0.6576	0.6453	0.6329	0.6205	0.6080	0.5955	0.5830
11	0.7932	0.7832	0.7730	0.7626	0.7520	0.7412	0.7303	0.7193	0.7081	0.6968
12	0.8684	0.8607	0.8529	0.8448	0.8364	0.8279	0.8191	0.8101	0.8009	0.7916
13	0.9210	0.9156	0.9100	0.9042	0.8981	0.8919	0.8853	0.8786	0.8716	0.8645
14	0.9552	0.9517	0.9480	0.9441	0.9400	0.9357	0.9312	0.9265	0.9216	0.9165
15	0.9760	0.9738	0.9715	0.9691	0.9665	0.9638	0.9609	0.9579	0.9546	0.9513
16	0.9878	0.9865	0.9852	0.9838	0.9823	0.9806	0.9789	0.9770	0.9751	0.9730
17	0.9941	0.9934	0.9927	0.9919	0.9911	0.9902	0.9892	0.9881	0.9870	0.9857
18	0.9973	0.9969	0.9966	0.9962	0.9957	0.9952	0.9947	0.9941	0.9935	0.9928
19	0.9988	0.9986	0.9985	0.9983	0.9980	0.9978	0.9975	0.9972	0.9969	0.9965
20	0.9995	0.9994	0.9993	0.9992	0.9991	0.9990	0.9989	0.9987	0.9986	0.9984
21	0.9998	0.9998	0.9997	0.9997	0.9996	0.9996	0.9995	0.9995	0.9994	0.9993
22	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9998	0.9997	0.9997
23	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000