Chapter 9

Relations

Relations and Their Properties

Section 9.1

Section Summary

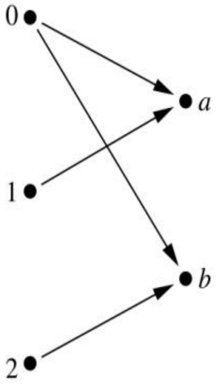
- Relations and Functions
- Properties of Relations
  - *Reflexive Relations*
  - *Symmetric and Antisymmetric Relations*
  - *Transitive Relations*
- Combining Relations

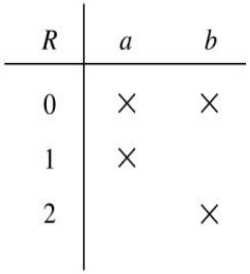
Binary Relations

Definition: A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.

Example:

- *Let A = {0,1,2} and B = {a,b}*
- *{(0,a), (0,b), (1,a) , (2,b)} is a relation from A to B.*
- *We can represent relations from a set A to a set B graphically or using a table:*



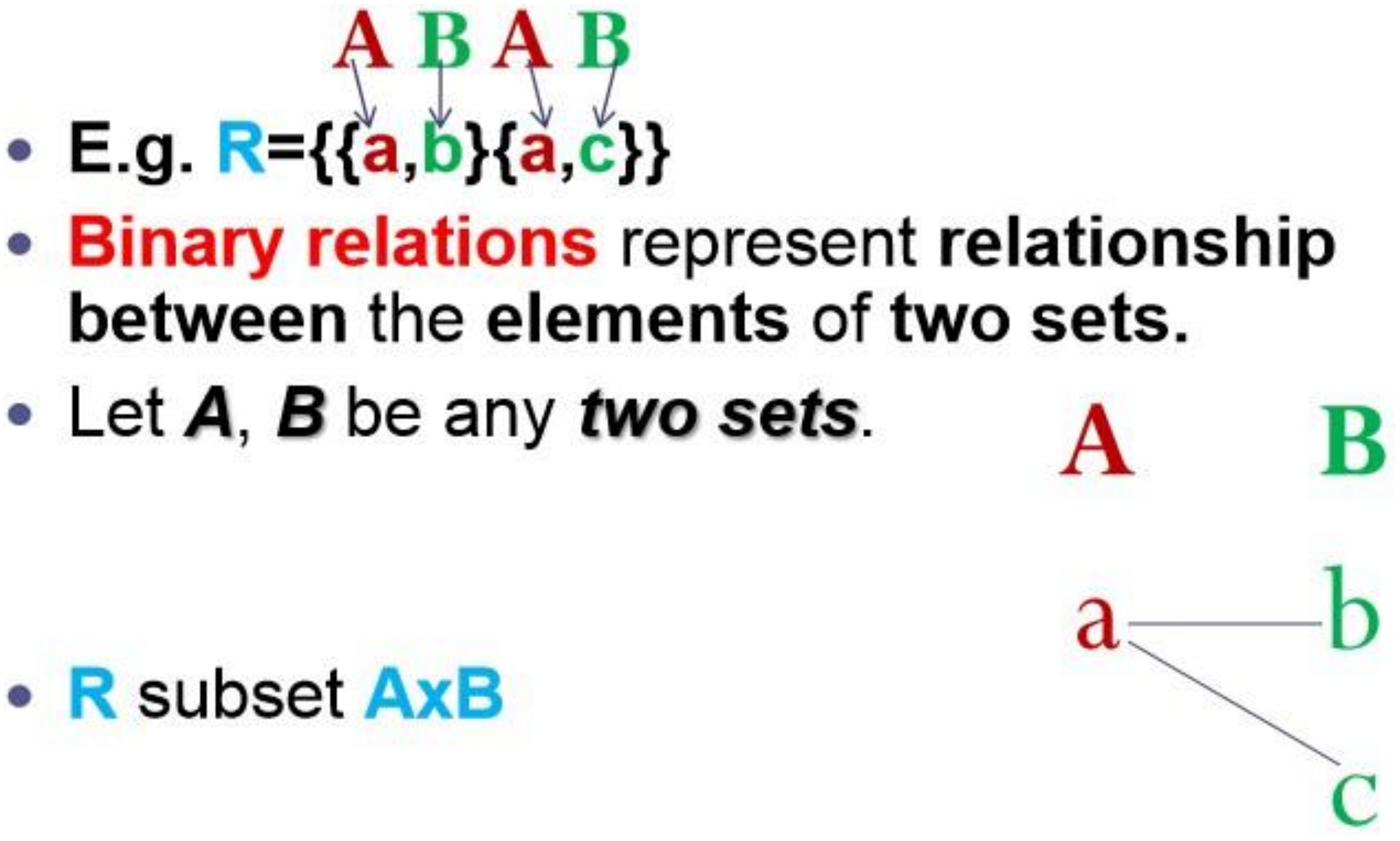


R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of *B* is related to each element of *A*.

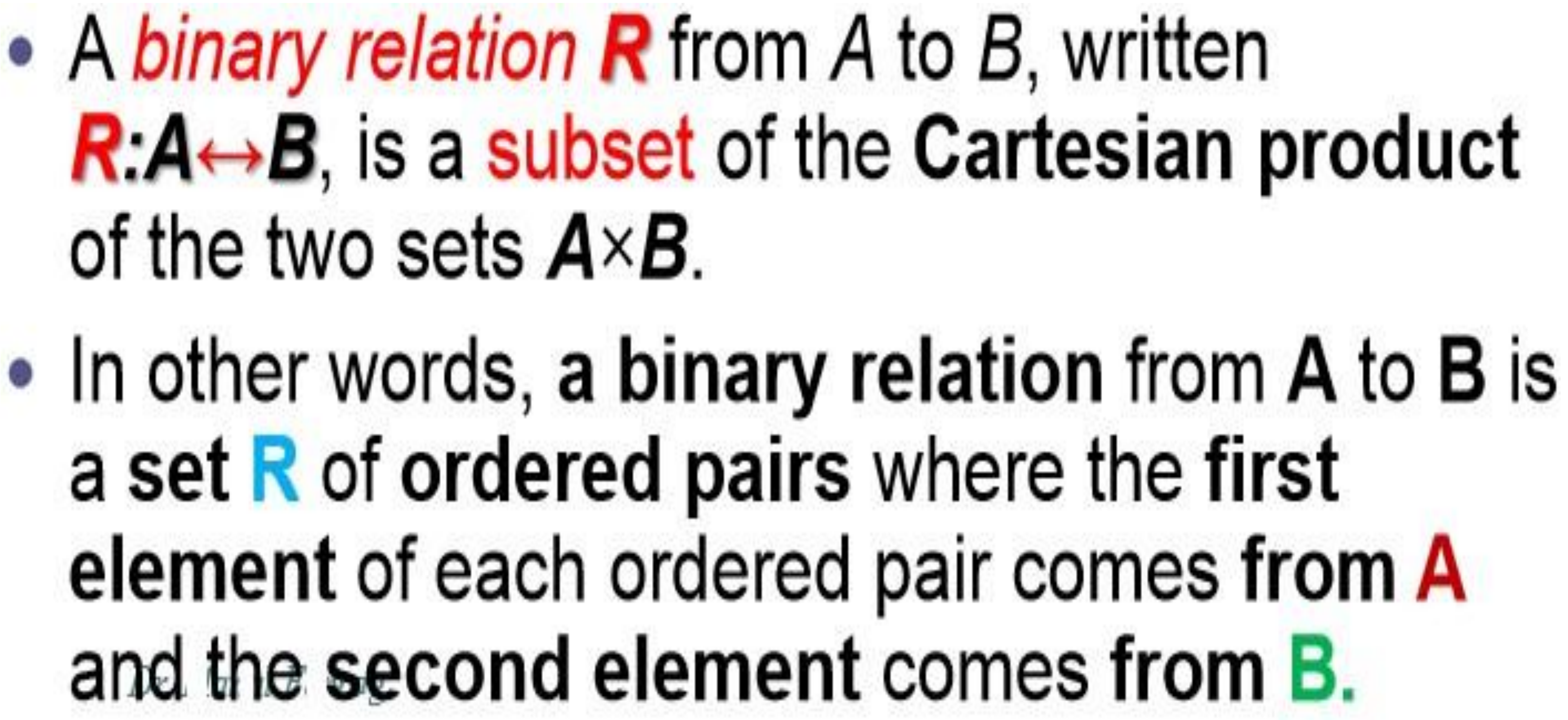
Relations and their properties

- **Sets of ordered pairs are called Binary relations**



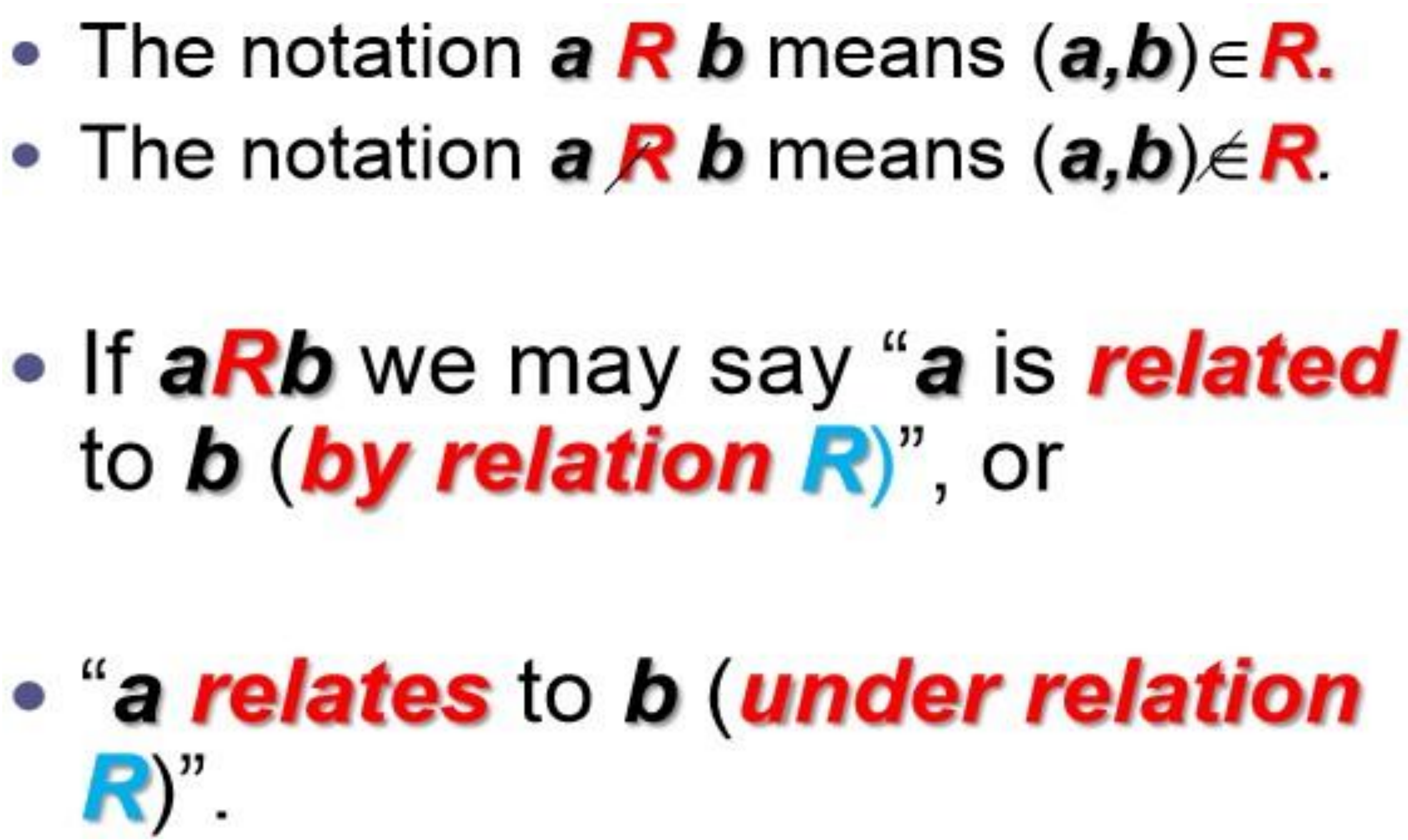
- E.g. R={{a,b}{a,c}}
- **Binary relations** represent **relationship between** the **elements** of **two sets.**
- Let ***A***, ***B*** be any ***two sets***.
- R subset AxB

Relations and their properties



- A *binary relation* ***R*** from *A* to *B*, written ***R*:*A*↔*B***, is a subset of the **Cartesian product** of the two sets ***A*×*B***.
- In other words, **a binary relation** from **A** to **B** is **a set** R of **ordered pairs** where the **first element** of each ordered pair comes **from A** and the **second element** comes **from** B.

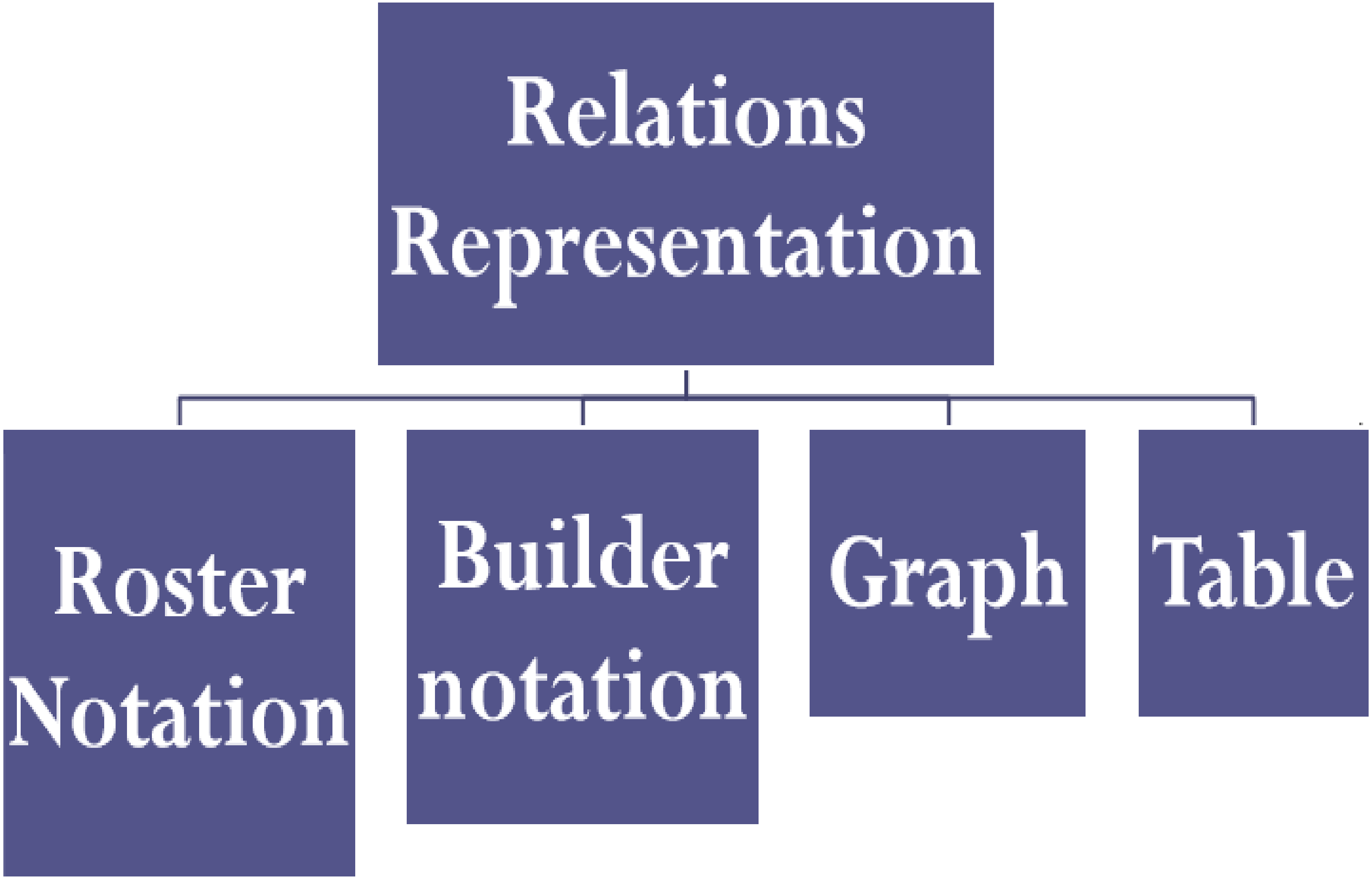
Relations and their properties



- The notation $a\ R\ b$ means $(a,b) \in R$.
- The notation $a \not{R}\ b$ means $(a,b) \notin R$.

- If aRb we may say "**a** is ***related*** to **b** (***by relation R***)", or

- "**a** ***relates*** to **b** (***under relation R***)".



Relations Representation

- Roster Notation
- Builder notation
- Graph
- Table

Roster notation

- Roster notation is **a list of elements, separated by commas, enclosed in curly braces.**
- Example: **{3, 5, 7}** is the set of single-digit odd prime numbers.
- Example: A = **{1,2,3,4,5}**

Builder notation

Builder notation gives **a rule** to **follow** that will tell you **how to build the roster**

A = {x | 0<x<6, x is a whole number}

Relations can be represented by:

A. **Roster Notation:**

List of ordered pairs

Ex: the set **{1,2,3,4}**

B. Set builder notation

Ex: R={(a,b)| a divides b}

Relations can be represented by:

Ex: Let **A** be the set **{1,2,3,4}**, which **ordered pairs** are in the relation **R={(a,b)| a divides b}**?

Graph notation

R1 = {(1,1),(2,2),(3,3),(4,4),(1,3),(1,2),
(1,4),(2,4)}

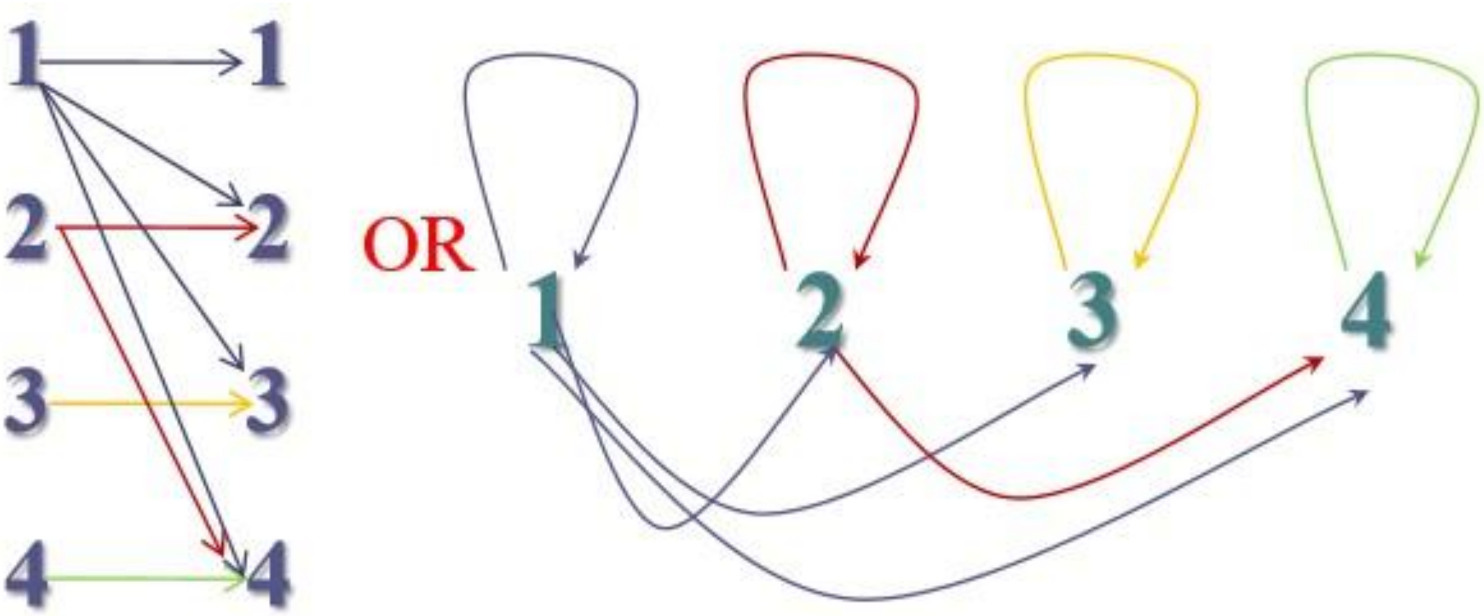


Table notation

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,2),(1,4),(2,4)\}$$

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Example

Let $R: A \to B$

$A = \{1, 2, 3\}$ represents **students**

$B = \{a, b\}$ represents **courses**

Example

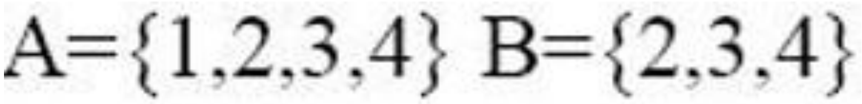
$A \times B =$

$\{ (1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$R = \{ (1,a), (1,b) \} \Rightarrow$ it means that **student 1** registered in **courses a** and **b**

Relations

Example: R: $A \to B$ such that (a>b)

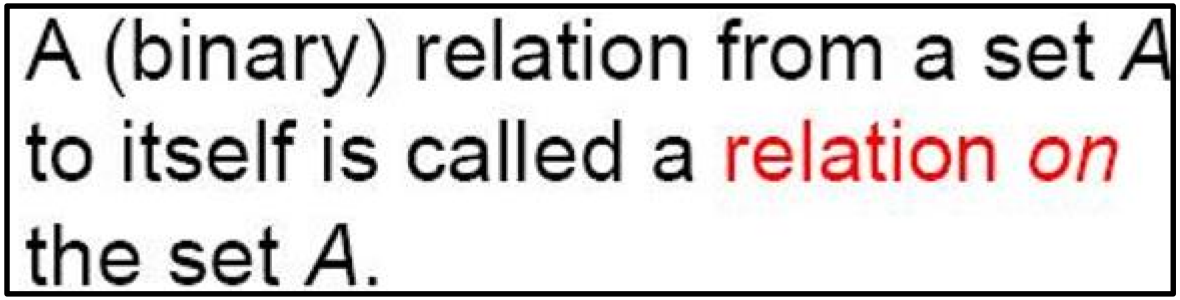


A={1,2,3,4} B={2,3,4}

Solution:

{(3,2) , (4,2), (4,3)}

Relations on Set



A (binary) relation from a set *A* to itself is called a relation *on* the set *A*.

Example

- Consider these relations on the set of integers:
- R1 $=\{(a,b)|a\leq b\}$,
- R2 $=\{(a,b)|a>b\}$,
- R3 $=\{(a,b)|a=b \text{ or } a=-b\}$,
- R4 $=\{(a,b)|a=b\}$,
- R5 $=\{(a,b)|a=b+1\}$,
- R6 $=\{(a,b)|a+b\leq 3\}$.

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Example

Solution:

- (1,1) R1, R3, R4, R6
- (1,2) R1, R6
- (2,1) R2, R5, R6
- (1,-1) R2, R3, R6
- (2,2) R1, R3, R4

Identity relation $\mathbf{I}_A$

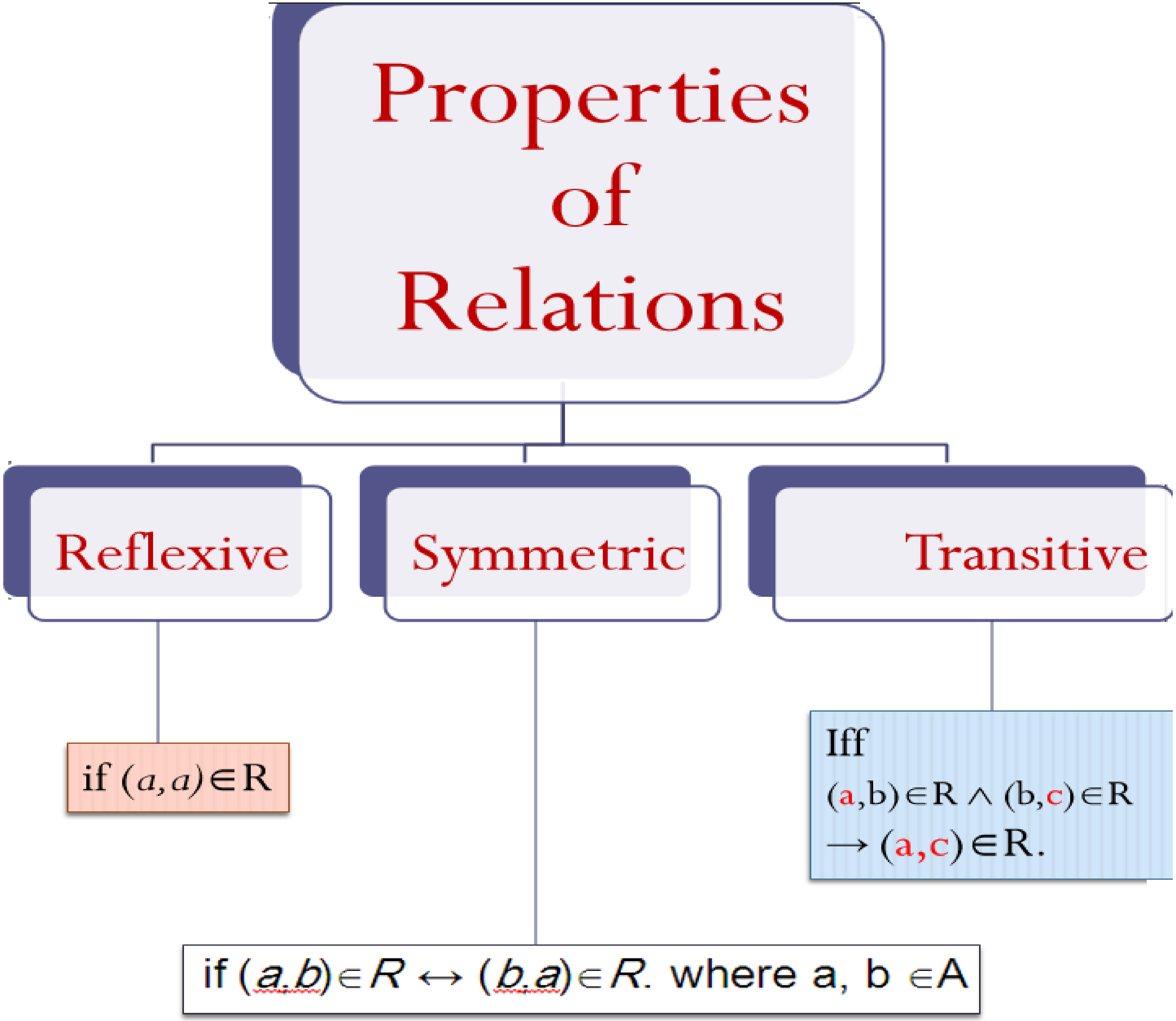
ε

The *identity relation* $\mathbf{I}_A$ on a set A is the set $\{(a,a) | a \in A\}$.

Example:

$A = \{1,2,3,4\}$

$\mathbf{I}_A = \{(1,1),(2,2),(3,3),(4,4)\}$



Properties of Relations

Reflexive

if $(a,a) \in R$

Symmetric

if $(a,b) \in R \leftrightarrow (b,a) \in R$. where a, b $\in A$

Transitive

Iff

$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$.

1. Reflexivity & Irreflexivity

A relation *R* on *A* is *reflexive* if $(a,a) \in R$
for every element $a \in A$.

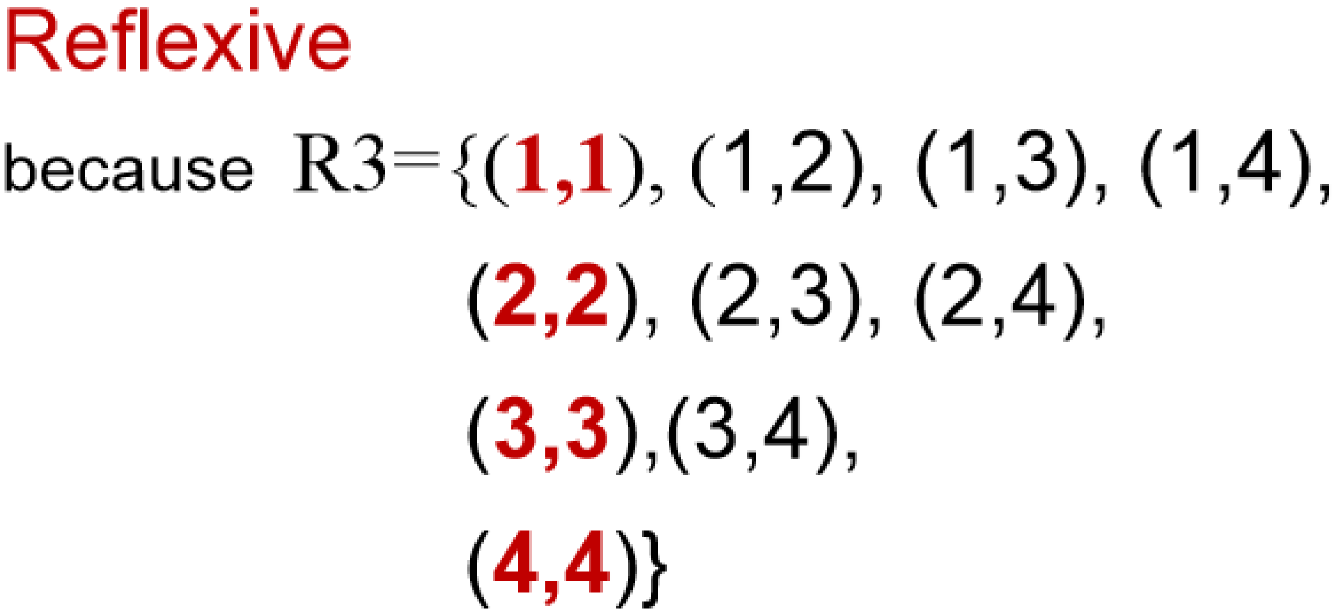
Ex: Consider the following relations on {1,2,3,4}

1. **R1 = {(1,1), (1,2), (2,1), (2,2) , (3,4), (4,1), (4,4)}**

2. **R2={(1,1), (2,1), (2,2), (3,3), (3,4),(4,4) }**

3. **R3 = {(a, b) | a ≤ b}**

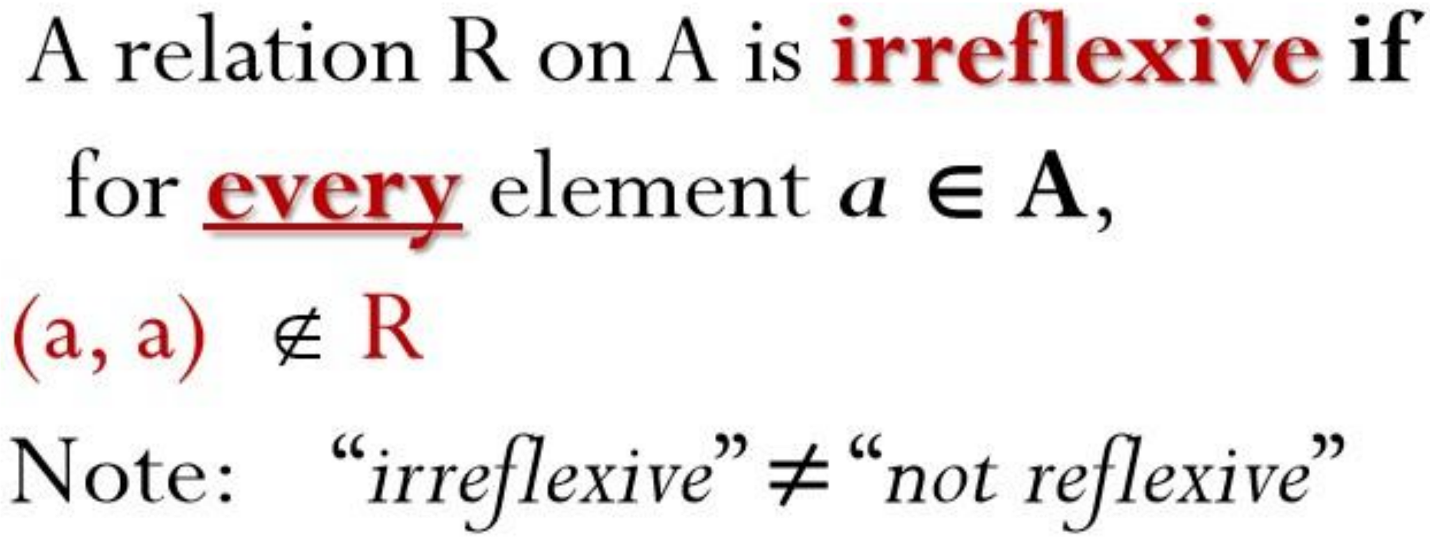
1. Reflexivity & Irreflexivity – cont.



Reflexive

because R3={(**1,1**), (1,2), (1,3), (1,4),
(**2,2**), (2,3), (2,4),
(**3,3**),(3,4),
(**4,4**)}

Difference between
Not Reflexive and Irreflexive



A relation R on A is **irreflexive** if

for **every** element $a \in \mathbf{A}$,

(a, a) $\notin$ R

Note: “*irreflexive*” ≠ “*not reflexive*”

Difference between
Not Reflexive and Irreflexive

Example:

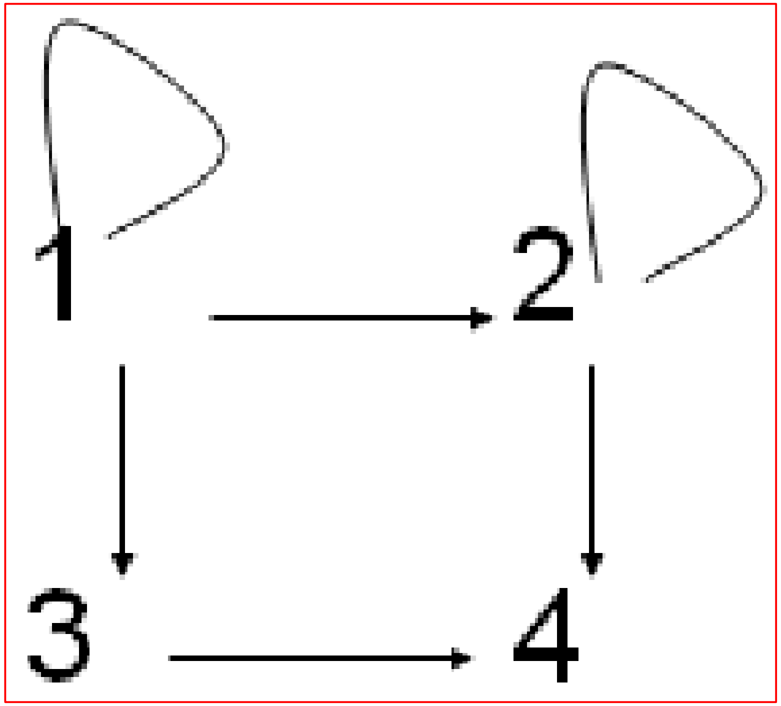
A= {1, 2}

R = {(1, 2), (2, 1), (1, 1)}

Note: "irreflexive" ≠ "not reflexive"

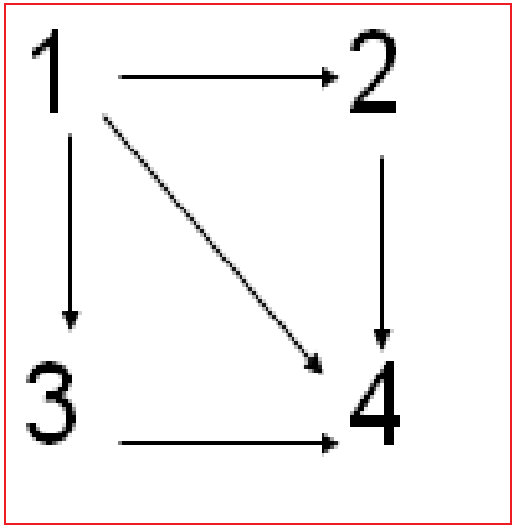
Reflexive or not?

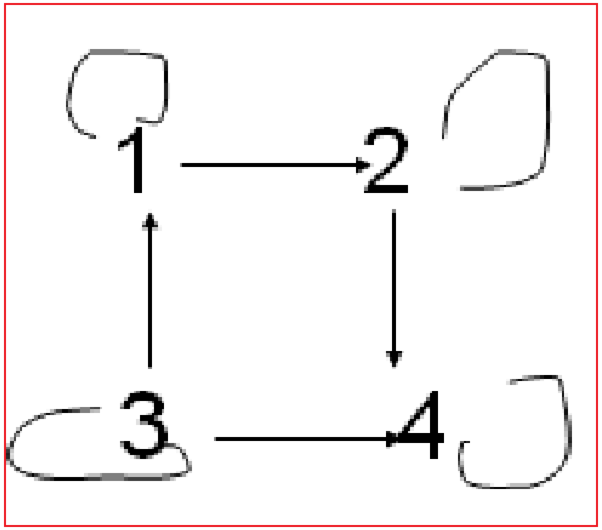
Irreflexive or not?



Not Reflexive and Not Irreflexive

Reflexive or not?
Irreflexive or not?

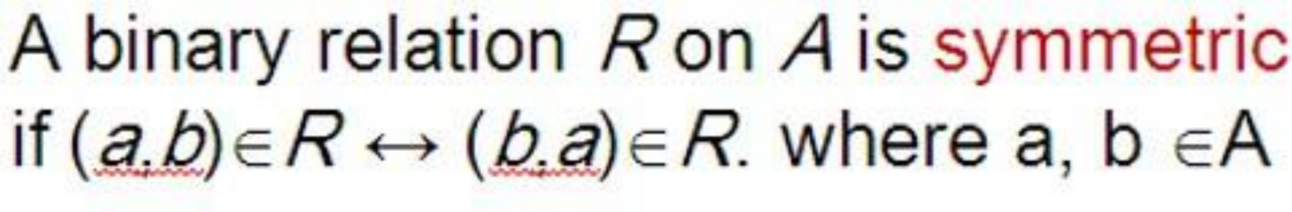




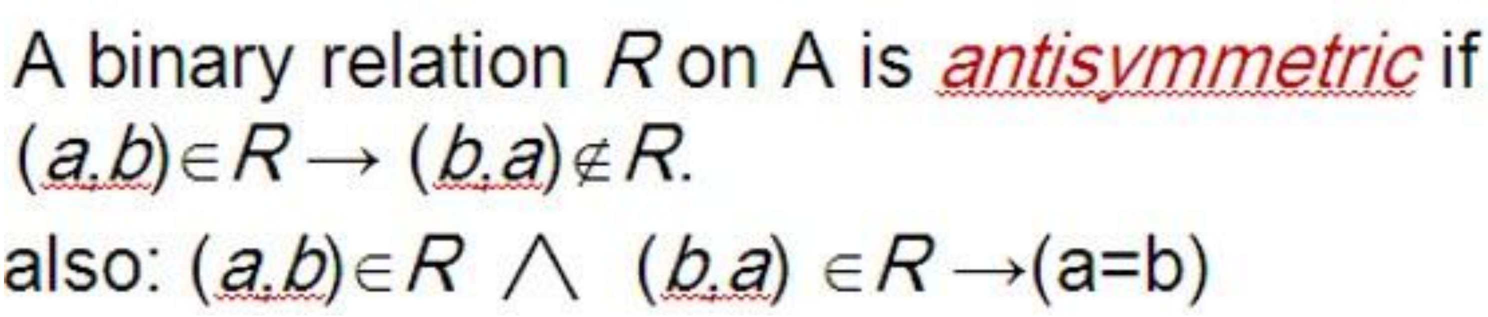
Irreflexive

Reflexive

2. Symmetry & Antisymmetry



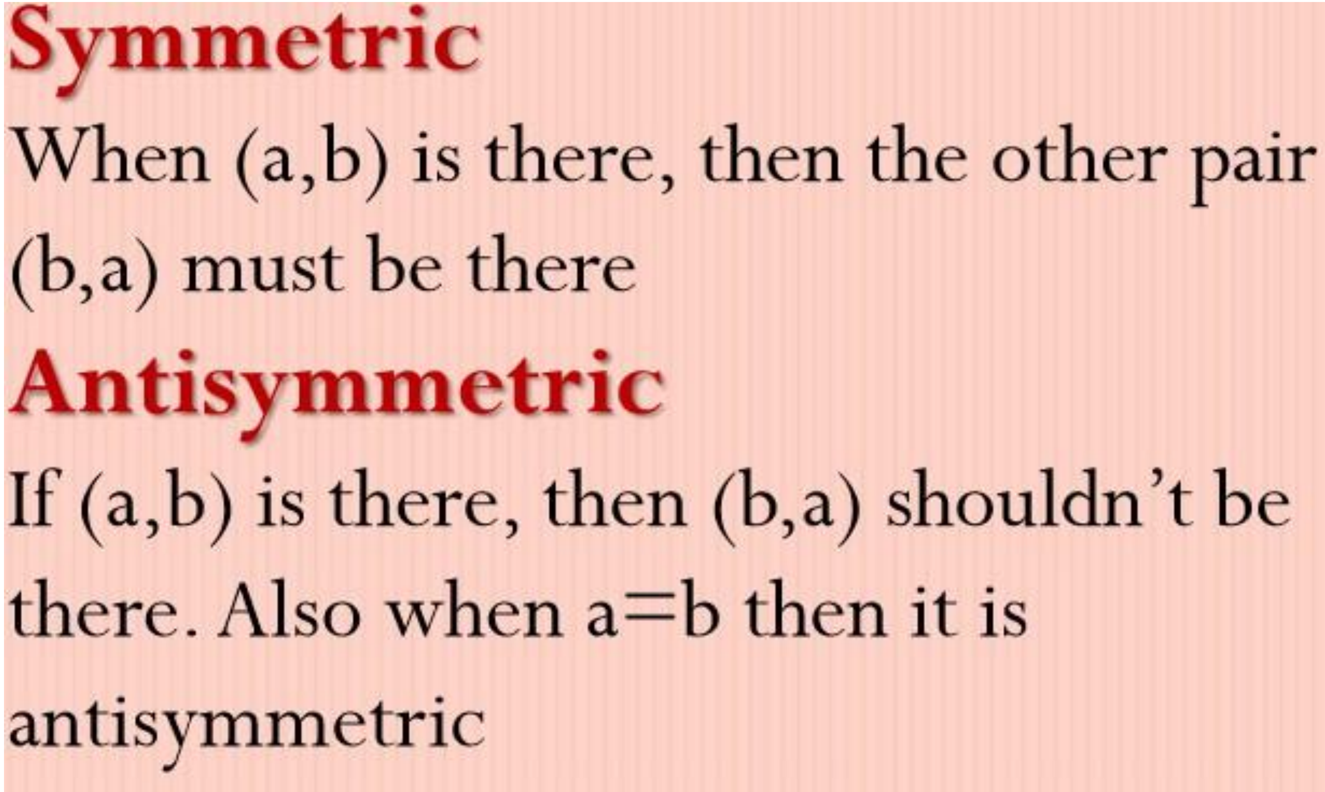
A binary relation R on A is symmetric if $(a.b)\in R \leftrightarrow (b.a)\in R$. where a, b $\in A$



A binary relation R on A is *antisymmetric* if $(a.b)\in R \rightarrow (b.a)\notin R$.

also: $(a.b)\in R \ \wedge \ (b.a) \in R \rightarrow (a=b)$

2. Symmetry & Antisymmetry



Symmetric

When (a,b) is there, then the other pair (b,a) must be there

Antisymmetric

If (a,b) is there, then (b,a) shouldn't be there. Also when a=b then it is antisymmetric

Reflexive or not Irreflexive or not Symmetric or not Antisemmetric or not?

Let A = {1,2,3}

R1={(1,2),(2,2),(3,1),(1,3)}	not reflexive, not irreflexive, not symmetric, not antisymmetric

Reflexive or not Irreflexive or not Symmetric or not Antisemmetric or not?

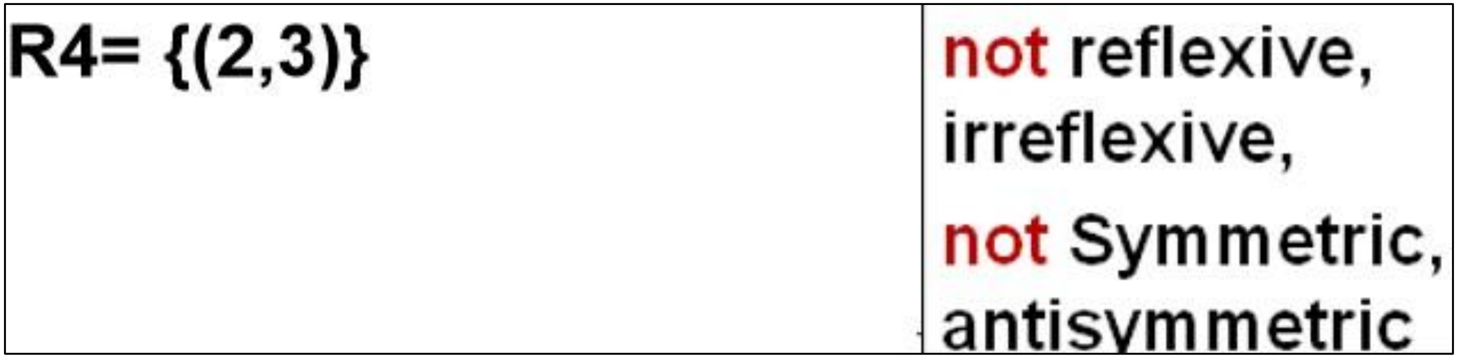
Let A = {1,2,3}

R2={(2,2),(1,3),(3,2)}	not reflexive, not irreflexive, not symmetric, Antisymmetric

R3 = {(1,1),(2,2),(3,3)}	reflexive, not irreflexive, symmetric, antisymmetric

Reflexive or not Irreflexive or not Symmetric or not Antisemmetric or not?

Let A = {1,2,3}



R4= {(2,3)}	not reflexive, irreflexive, not Symmetric, antisymmetric

Symmetric or not
Antisemmetric or not?

Consider these relations on the set of integers:

R1={(a, b) | a=b}

symmetric , antisymmetric

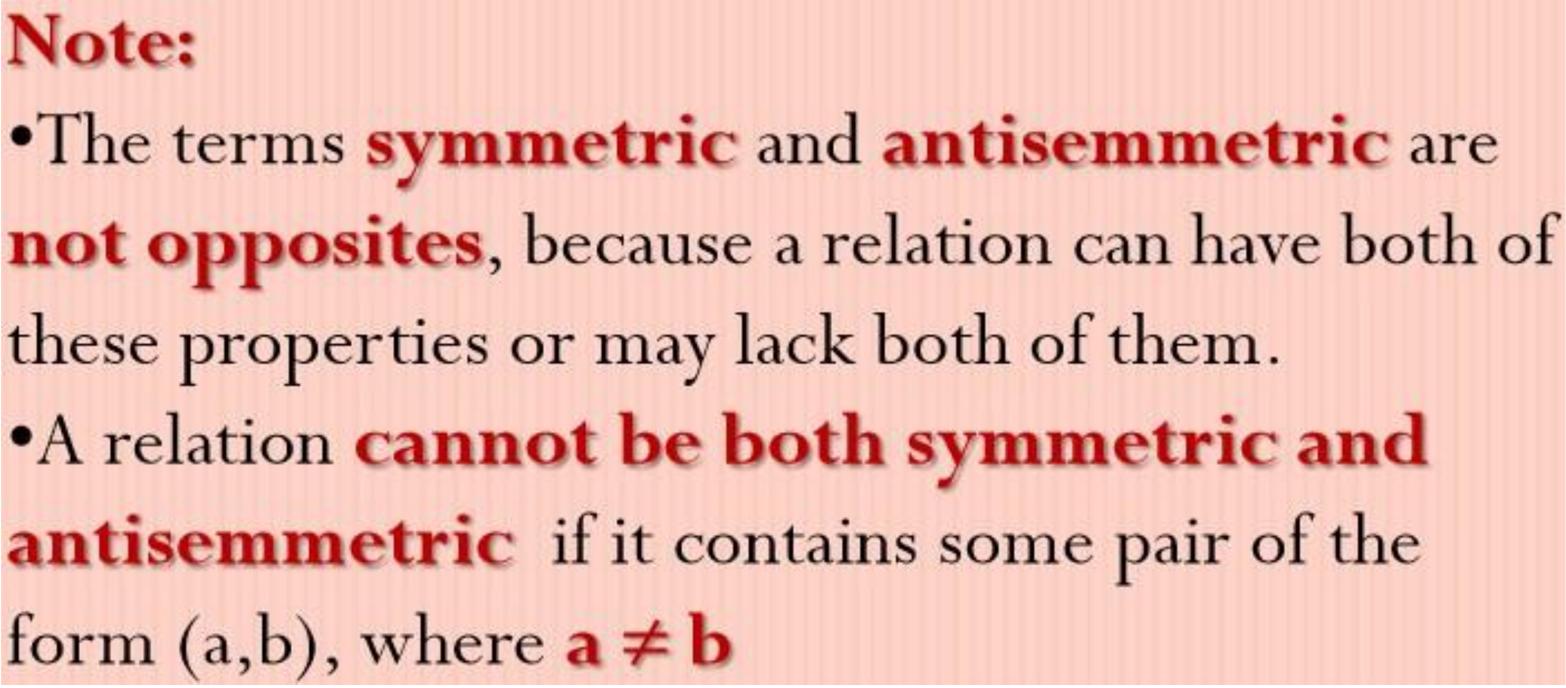
R2={(a, b) | a>b}

not symmetric, antisymmetric

R3={(a, b) | a=b+1}

not symmetric, antisymmetric

Symmetric or not Antisemmetric or not?



Note:

- The terms **symmetric** and **antisemmetric** are **not opposites**, because a relation can have both of these properties or may lack both of them.
- A relation **cannot be both symmetric and antisemmetric** if it contains some pair of the form (a,b), where **a ≠ b**

3. Transitivity

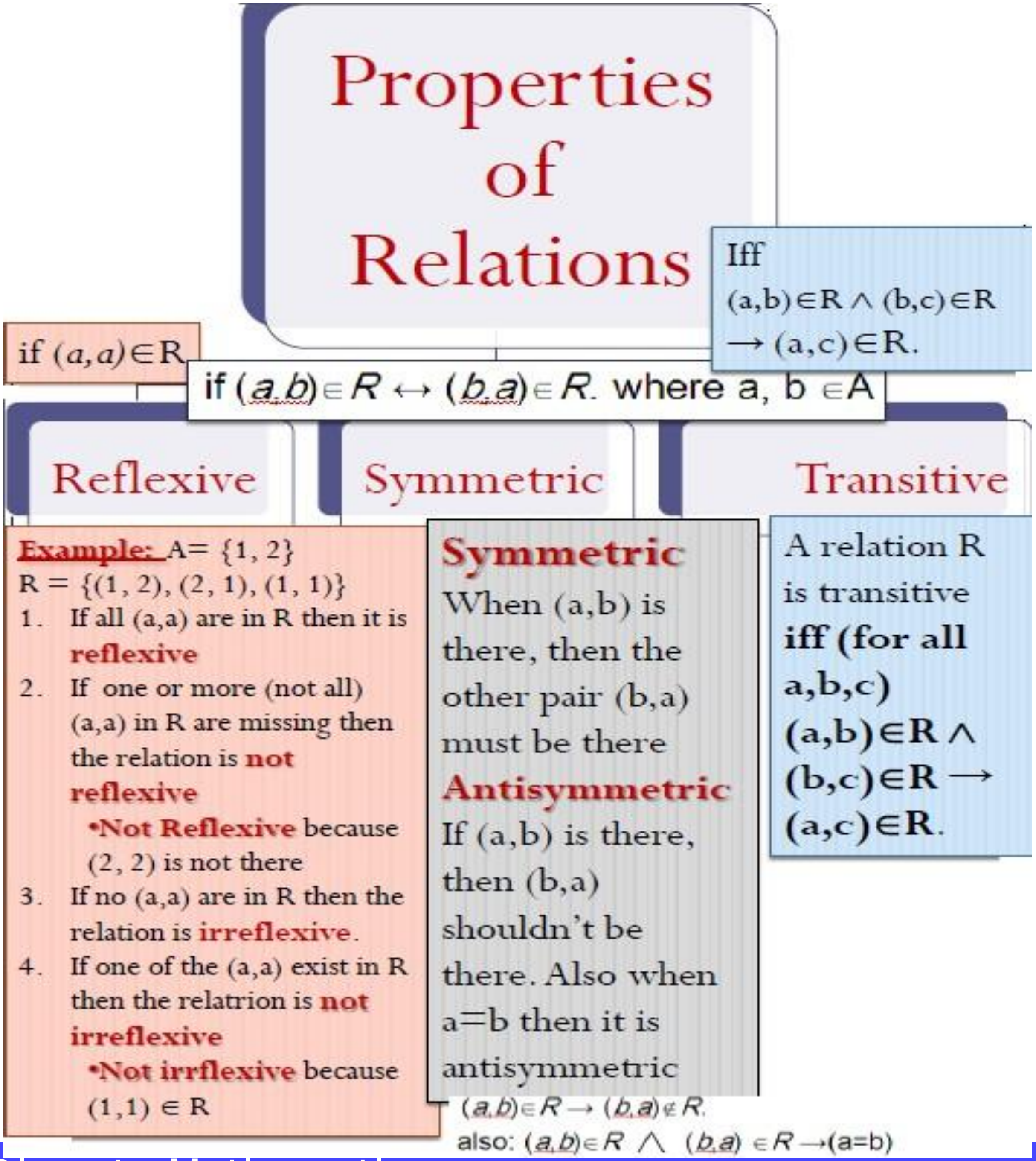
- A relation R is *transitive* iff (for all a,b,c) $(a,b) \in R \land (b,c) \in R \rightarrow (a,c) \in R$.

Examples:

A= {1,2}

R1={(1,1),(1,2),(2,1),(2,2)} transitive

R2={(1,1),(1,2),(2,1)} not transitive, (2,2) $\notin$R2



Properties of Relations

if $(a,a) \in R$

if $(a,b) \in R \leftrightarrow (b,a) \in R$. where a, b $\in A$

Iff
$(a,b) \in R \wedge (b,c) \in R$
$\rightarrow (a,c) \in R$.

Reflexive

Example: A= {1, 2}
R = {(1, 2), (2, 1), (1, 1)}

1. If all (a,a) are in R then it is **reflexive**
2. If one or more (not all) (a,a) in R are missing then the relation is **not reflexive**
   - **Not Reflexive** because (2, 2) is not there
3. If no (a,a) are in R then the relation is **irreflexive**.
4. If one of the (a,a) exist in R then the relatrion is **not irreflexive**
   - **Not irrflexive** because (1,1) $\in$ R

Symmetric

Symmetric

When (a,b) is there, then the other pair (b,a) must be there

Antisymmetric

If (a,b) is there, then (b,a) shouldn't be there. Also when a=b then it is antisymmetric

$(a,b) \in R \rightarrow (b,a) \notin R$.

also: $(a,b) \in R \wedge (b,a) \in R \rightarrow (a=b)$

Transitive

A relation R is transitive **iff (for all a,b,c) $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$.**

Combining Relations

Let $A=\{1,2,3\}$, $B=\{1,2,3,4\}$

$R1=\{(1,1),(2,2),(3,3)\}$

$R2=\{(1,1),(1,2),(1,3),(1,4)\}$

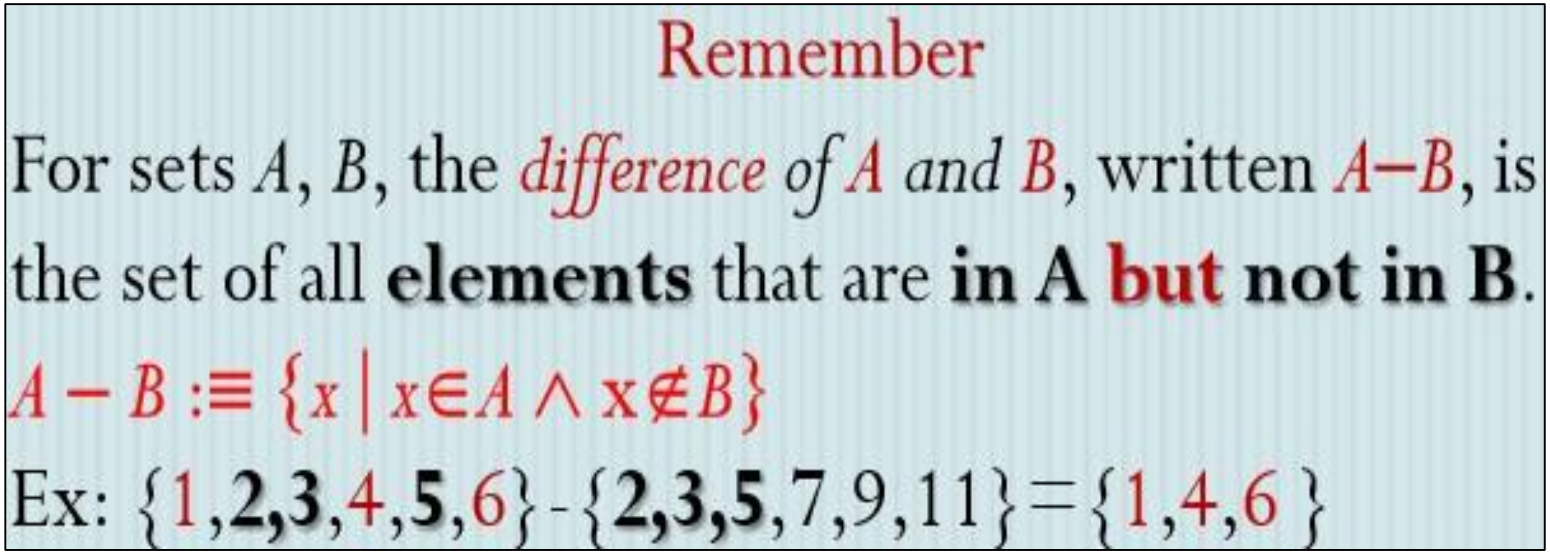
$R1 \cup R2 = \{(1,1),(2,2),(3,3),$
$(1,2),(1,3),(1,4)\}$

$R1 \cap R2= \{(1,1)\}$

$R1- R2=\{(2,2),(3,3)\}$

$R2-R1=\{(1,2),(1,3),(1,4)\}$

Combining Relations



Remember

For sets *A*, *B*, the *difference of A and B*, written *A*−*B*, is the set of all **elements** that are **in A but not in B**.

$A - B :\equiv \{x \mid x \in A \wedge x \notin B\}$

Ex: {1,**2**,**3**,4,**5**,6}-{**2**,**3**,**5**,7,9,11}={1,4,6 }

Composition

Definition: Suppose

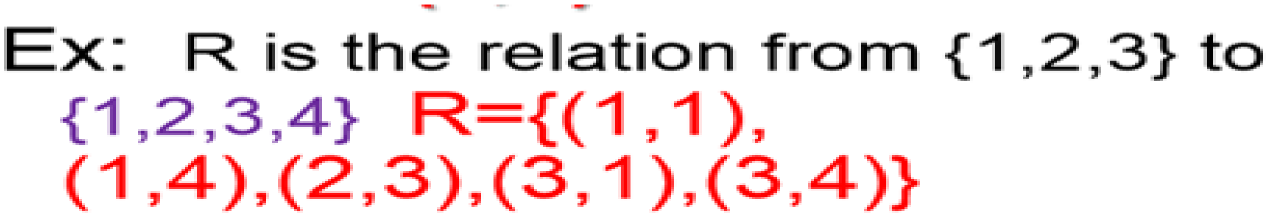
- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from *A* to *C* where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

- If (a,c) is in R1 and (c,b) is in R2 then (a,b) is in **R2 ∘ R1**

Composite Relations

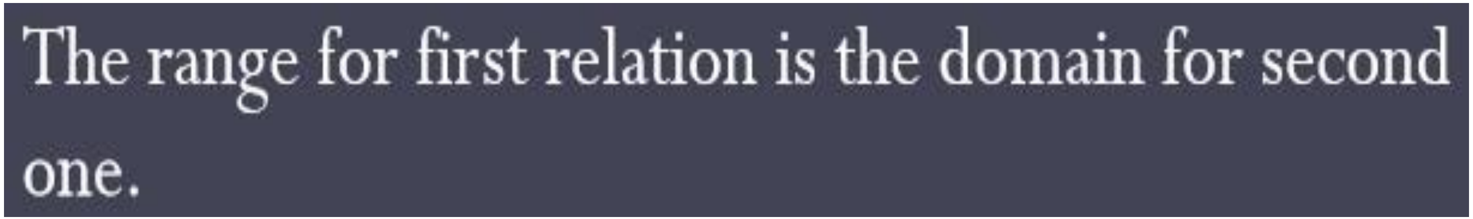


Ex: R is the relation from {1,2,3} to {1,2,3,4} R={(1,1), (1,4),(2,3),(3,1),(3,4)}

S is the relation from {1,2,3,4} to {0,1,2}

S = {{1,0), (2,0),(3,1),(3,2),(4,1)}

- S∘R = {(1,0),(1,1),(2,1),(2,2),(3,0),(3, 1)}



The range for first relation is the domain for second one.

Example

- Find $S \circ R$ if $R = \{(1,1), (1,2), (2,5), (3,4)\}$ and $S = \{(1,6), (2,1), (4,0)\}$

- $S \circ R = \{(1,6), (1,1), (3,0)\}$

Power

- Let R be a relation on the set A. the power R^n, n=1,2,3… are defined by

$R^1 = R$ and $R^{n+1} = R^n \circ R$

Ex: Let R ={(1,1),(2,1),(3,2),(4,3)}.

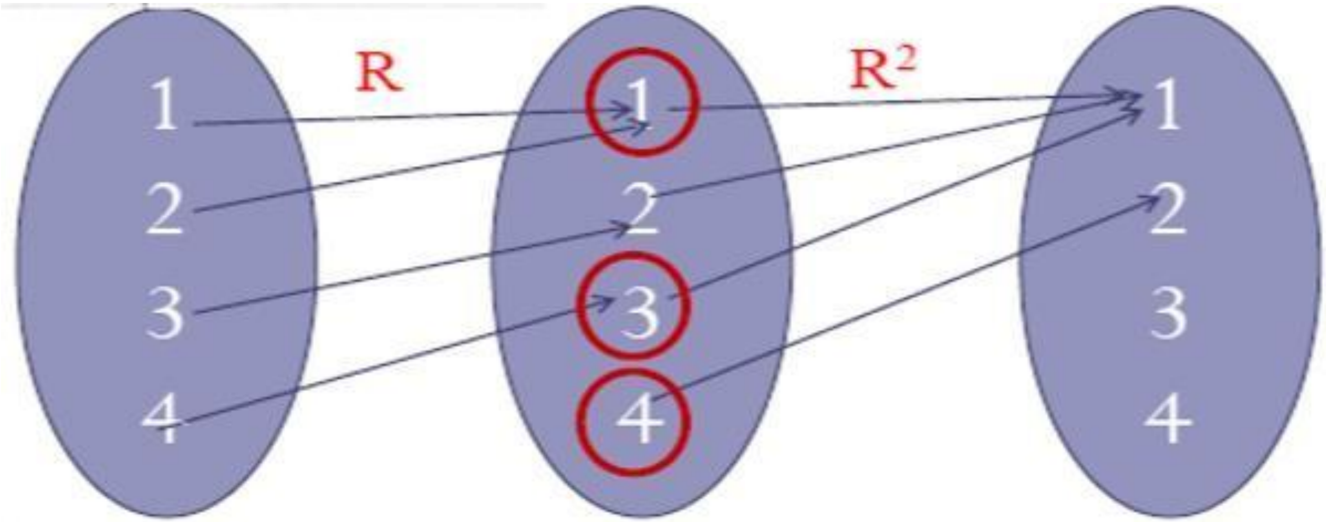
Find:

- $R^2 = R \circ R$= {(1,1),(2,1),(3,1),(4,2)}
- $R^3 = R^2 \circ R$= {(1,1),(2,1),(3,1),(4,1)}

Power

- Additional computations shows that R^4 is the same as R^3, so $R^4=R^3 \circ R= \{(1,1),(2,1),(3,1),(4,1)\}$
- It also follows that $R^n = R^3$ for $n=5,6,7,$.........Please verify this!

Example



R^3 = {(1,1),(2,1),(3,1),(4,1)}

Representing Relations

Section 9.3

Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

Representing Relations

Some special ways to represent binary relations:

- With **ordered pairs** (section 9.1)
- With **a table** (section 9.1)
- With a **Zero-one matrix**
- With a **directed graph** or **diagraph**

Using Zero-One Matrices

- To represent a **relation R** by a **matrix $\mathbf{M}_R$** = [m_{ij}], let $m_{ij} = 1$ if $(a_i, b_j) \in R$, else 0.
- *E.g.*, **Joe** likes **Susan** and **Mary**, **Fred** likes **Mary**, and **Mark** likes **Sally**.

Using Zero-One Matrices

- The **0-1 matrix** representation of that "**Likes**" **relation**:

$$\begin{array}{rc} & \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{array}{r} \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Using Zero-One Matrices

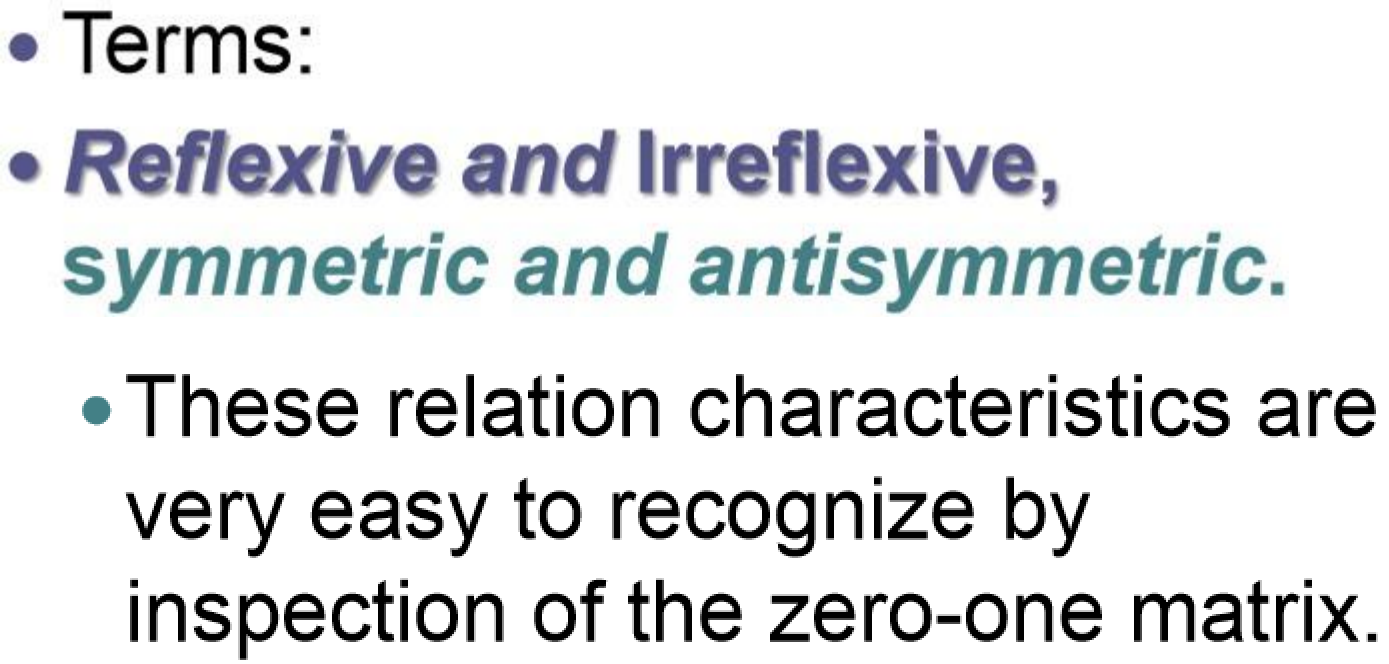
Example:

A={1,2,3} , B={1,2} , R: A↔B such that:

R= {(2,1),(3,1),(3,2)} (**a>b**) then the **matrix for R** is:

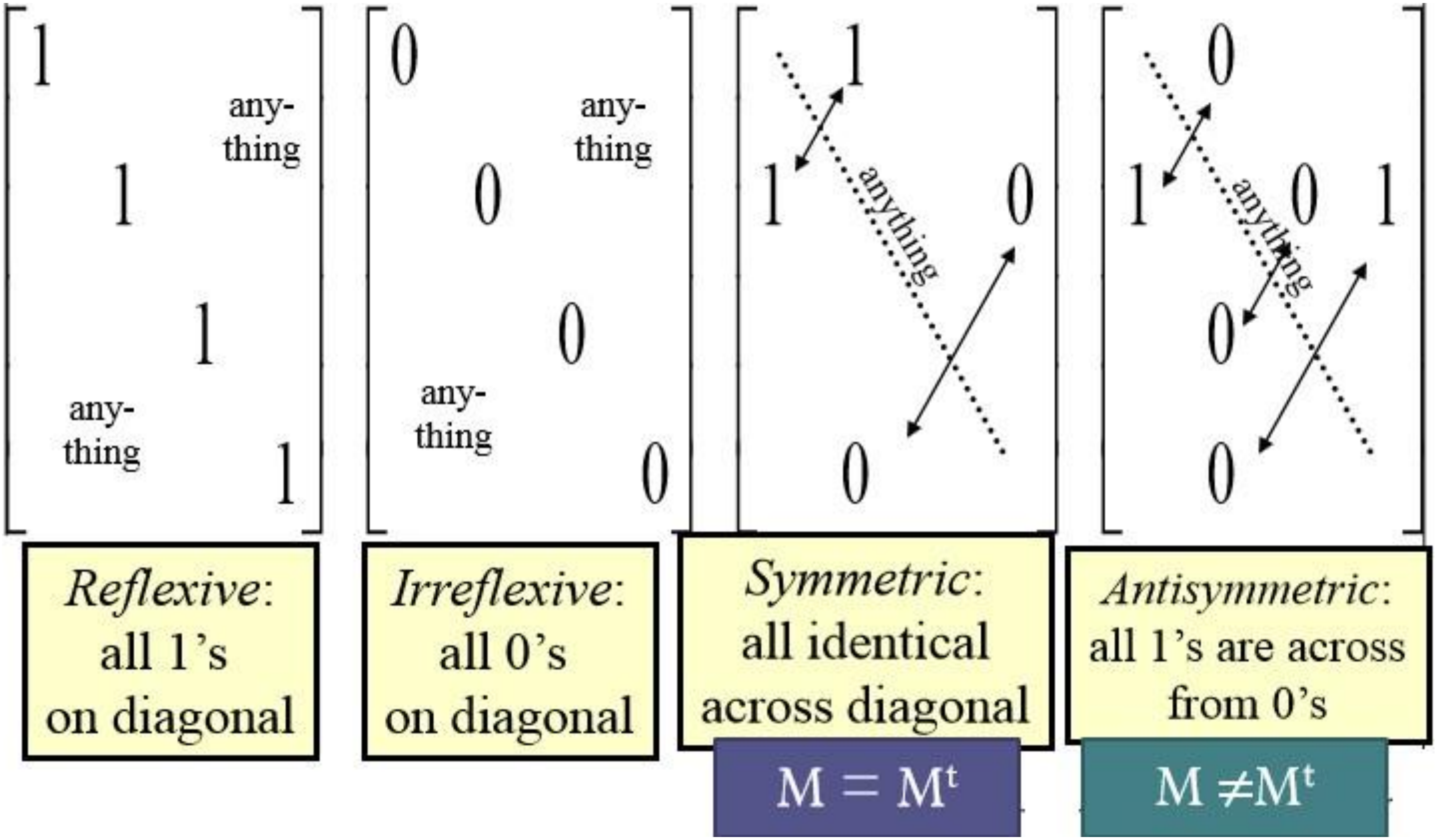
$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Zero-One Reflexive, Symmetric



- Terms:
- ***Reflexive and Irreflexive, symmetric and antisymmetric.***
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.

Zero-One Reflexive, Symmetric



Reflexive: all 1's on diagonal

Irreflexive: all 0's on diagonal

Symmetric: all identical across diagonal

$M = M^t$

Antisymmetric: all 1's are across from 0's

$M \neq M^t$

Example

Is **R** reflexive, symmetric, antisymmetric?

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Reflexive, symmetric, not antisymmetric

The Boolean Operations 1) join and meet

1) Join and meet representing Union and the Intersection

The Boolean Operations **join** $\vee$ and **meet** $\wedge$ **can be used** → to find the matrices representing the **union** $\cup$ and the **intersection** $\cap$ of two relations

Then:

$$M_{R1\cup R2} = M_{R1} \vee M_{R2}$$

$$M_{R1\cap R2} = M_{R1} \wedge M_{R2}$$

Example

Suppose that the relations **R1** and **R2** on set A are represented by the matrices:

$$M_{R1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad and \quad M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the **matrices** representing $R1 \cup R2$ and $R1 \cap R2$

Example

$$M_{R1\cup R2} = M_{R1} \vee M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R1\cap R2} = M_{R1} \wedge M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2) Composite

2) Composite

Suppose that **R**: A ↔ B, **S**: B ↔ C

(Boolean Product)

$$M_{S\circ R} = M_R \odot M_S$$

Remember how we did Boolean product!!!!!!!

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Remember how we did Boolean product!!!!!!!

$$\mathbf{A} \odot \mathbf{B} =$$

$$\begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Example

$$M_{S\circ R} = M_R \Theta M_S \neq M_S \Theta M_R$$

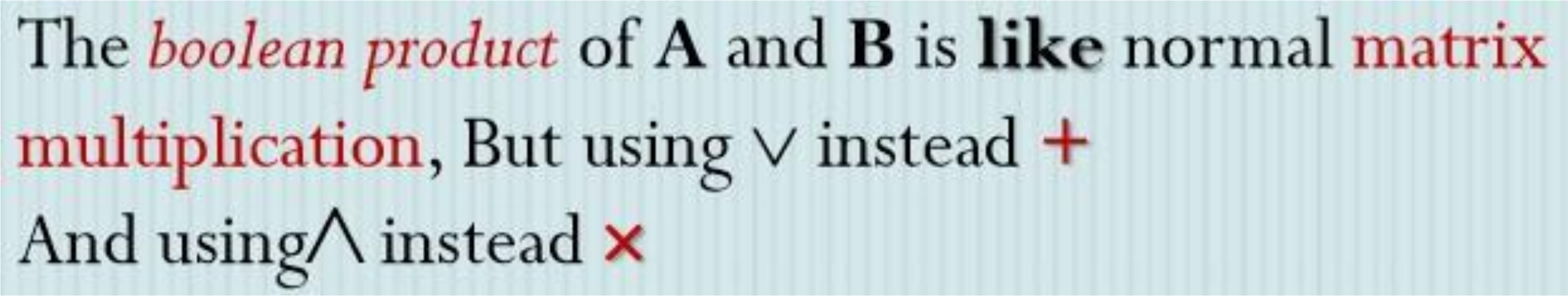
Let:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad and \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix of $S \circ R$?

Example

$$M_{S\circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



The *boolean product* of **A** and **B** is **like** normal matrix multiplication, But using ∨ instead +

And using∧ instead ×

Attention

$$M_{S \circ R} = M_R \Theta M_S \neq M_S \Theta M_R$$

$$Check!$$

$$M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad and \quad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Attention

$$M_{S\circ R} = M_S \Theta M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\neq M_R \Theta M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3) Power

$$M_{R^n} = M_R^{[n]}$$

- $R^2 = R \circ R \quad = M_R^{[2]}$
- $R^3 = R^2 \circ R \quad = M_R^{[3]}$

Example

- Find the matrix that represent **R^2** where the matrix representing **R** is:

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad then \quad M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Example

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad and \quad M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Find the matrix that represent **R^3** where the matrix representing **R** **and** **R^2** is:

$$M_{R^3} = M_{R^2} \Theta M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

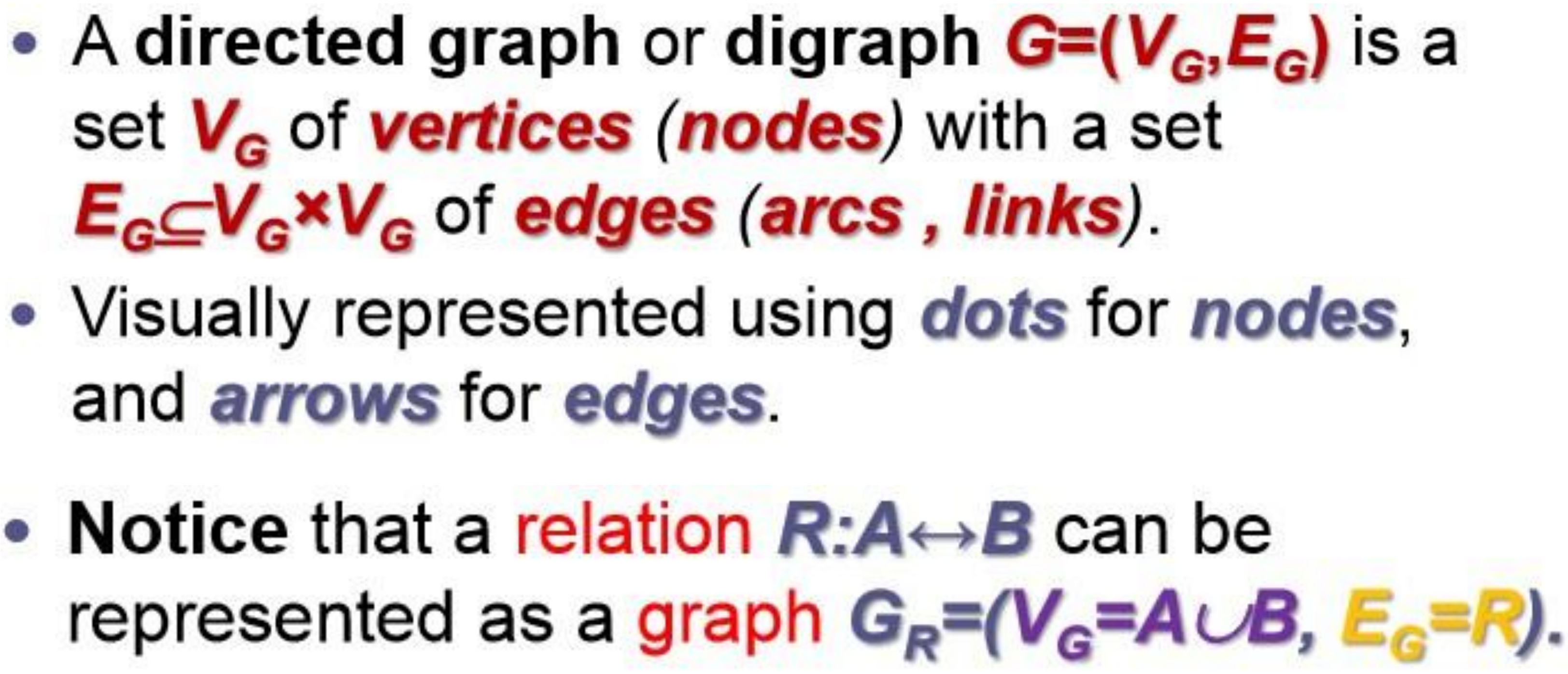
Example

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad and \quad M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= M_R \Theta M_{R^2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

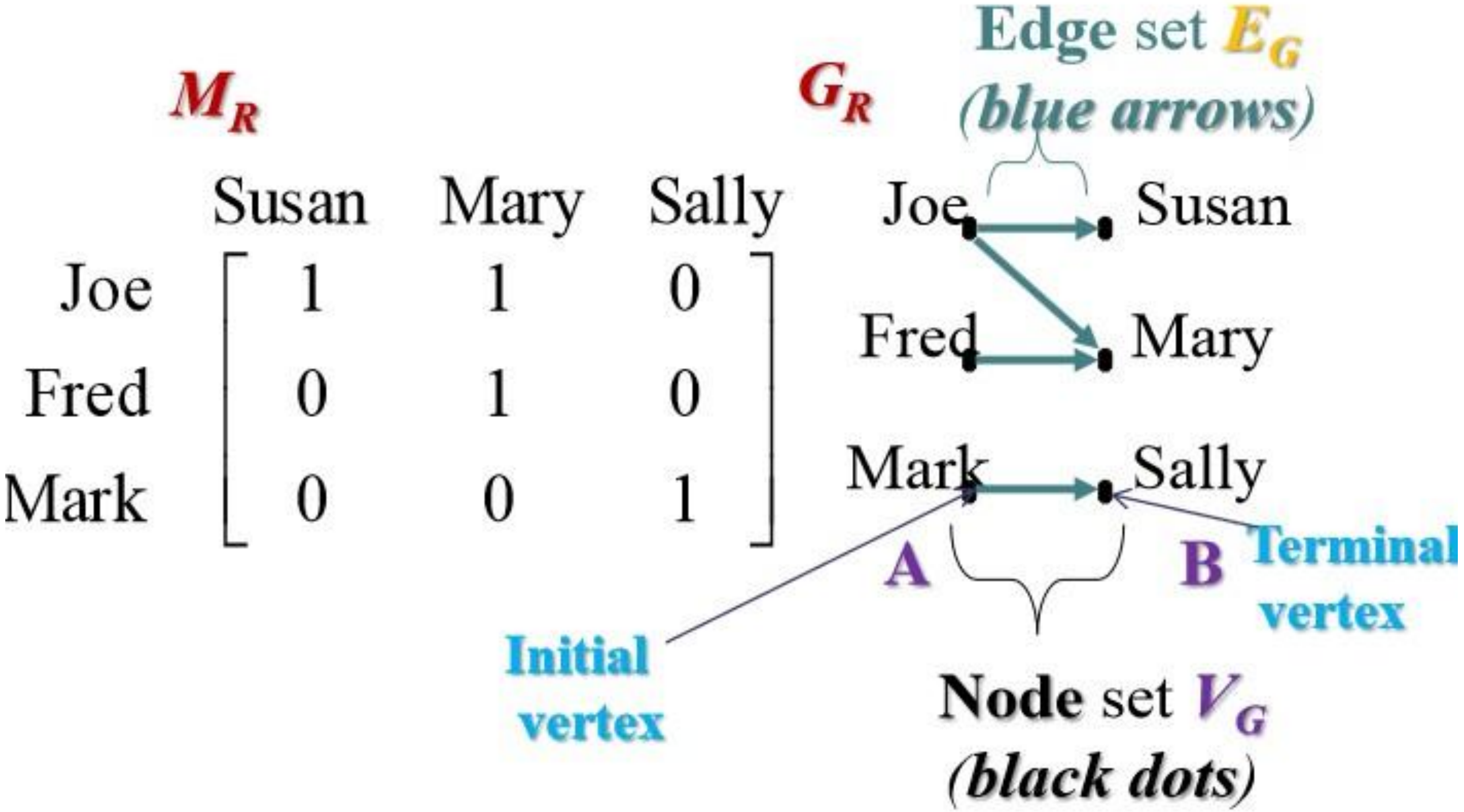
Using Directed Graphs



- A **directed graph** or **digraph** $G=(V_G,E_G)$ is a set V_G of ***vertices*** (***nodes***) with a set $E_G \subseteq V_G \times V_G$ of ***edges*** (***arcs*** , ***links***).
- Visually represented using ***dots*** for ***nodes***, and ***arrows*** for ***edges***.

- **Notice** that a relation $R:A \leftrightarrow B$ can be represented as a graph $G_R=(V_G=A \cup B,\ E_G=R)$.

Using Directed Graphs



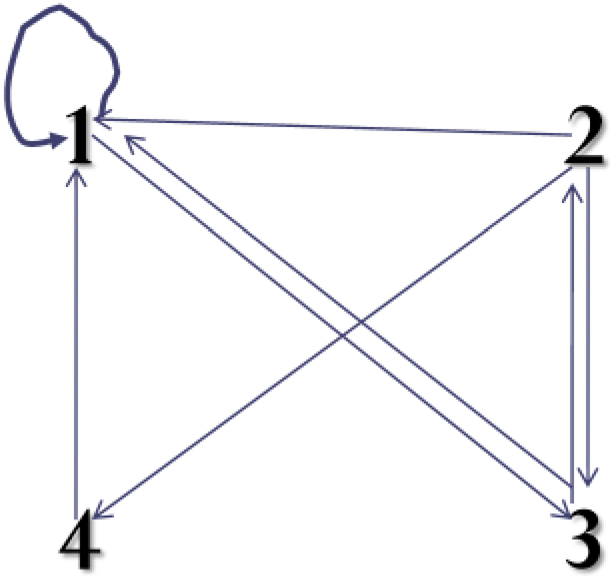
M_R

$$\begin{array}{c|ccc} & \text{Susan} & \text{Mary} & \text{Sally} \\ \hline \text{Joe} & 1 & 1 & 0 \\ \text{Fred} & 0 & 1 & 0 \\ \text{Mark} & 0 & 0 & 1 \end{array}$$

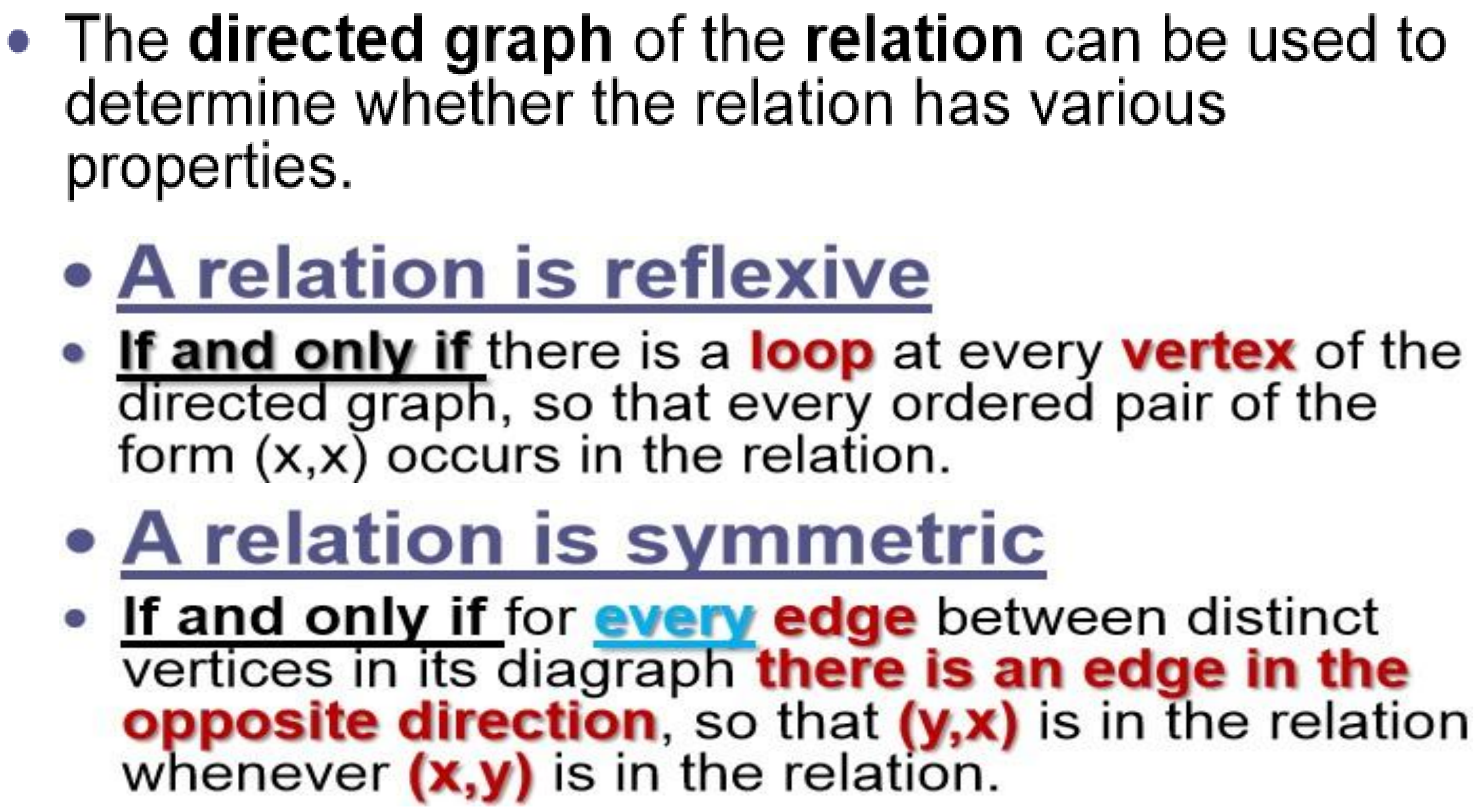
Example

- The directed graph of the relation
- R={(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2),(4,1)}

on the set {1,2,3,4} is



Directed graph and relation properties



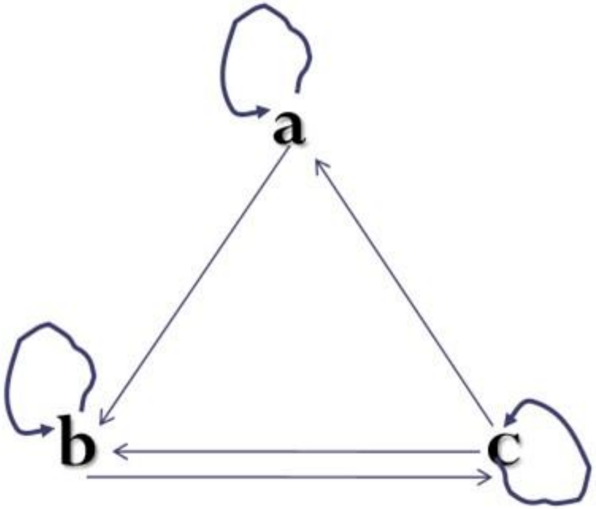
- The **directed graph** of the **relation** can be used to determine whether the relation has various properties.
  - **A relation is reflexive**
    - **If and only if** there is a **loop** at every **vertex** of the directed graph, so that every ordered pair of the form (x,x) occurs in the relation.
  - **A relation is symmetric**
    - **If and only if** for **every** **edge** between distinct vertices in its diagraph **there is an edge in the opposite direction**, so that **(y,x)** is in the relation whenever **(x,y)** is in the relation.

Directed graph and relation properties. cont

- **A relation is antisymmetric**
  - **If and only if** there are **never** **two edges in opposite directions** between distinct vertices.
- **A relation is transitive**
  - **If and only if** whenever there is an edge from a vertex **x** to a vertex **y** and an edge from vertex **y** to a vertex **z** , there is an edge from **x** to **z**. (completing a **triangle** where each side is a directed edge with the **correct direction**.)

Example

Determine whether the relations for the directed graph are reflexive, symmetric, antisymmetric, and/or transitive



Reflexive: because there are loops at every vertex.

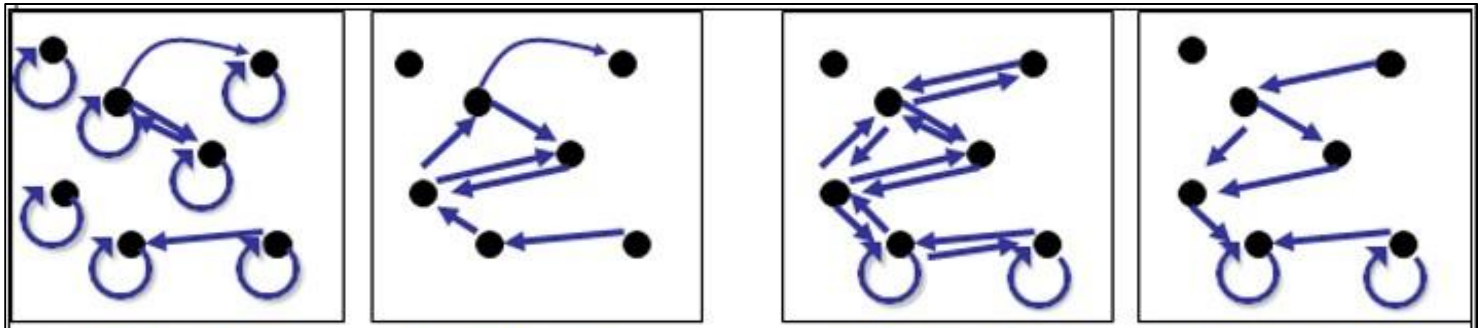
Not symmetric: because there is an edge from a to b and there is no edge from b to a.

Not antisymmetric: because there are edges in opposite directions connecting b and c.

Not transitive: because there is an edge from a to b and an edge from b to c , but no edge from a to c.

Digraph Reflexive, Symmetric

It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop

Irreflexive: No node links to itself

Symmetric: Every link is bidirectional

Antisymmetric: No link is bidirectional

Not symmetric, non-antisymmetric

Non-reflexive, non-irreflexive

Closure Relations

Section 9.4

1) Reflexive Closure

- For any property *X*, the **"X closure"** of a set *A* is defined as the "**smallest**" superset of *A* that has the given property.
- 1)The **reflexive closure** of a relation *R* on *A* is obtained →by adding **(a,a)** to **R** for each **$a \in A$** *not already in* **R**

I.e., it is $\mathbf{R} \cup \mathbf{I}_\mathbf{A}$

Example 1

The relation **R={(1,1),**(1,2),(2,1),(3,2)} on the set A ={1,2,3} is not reflexive.

How can you produce a reflexive relation containing R that is as small as possible?

Answer: By adding **(2,2)** and **(3,3)** so the :

Reflexive closure of R is:

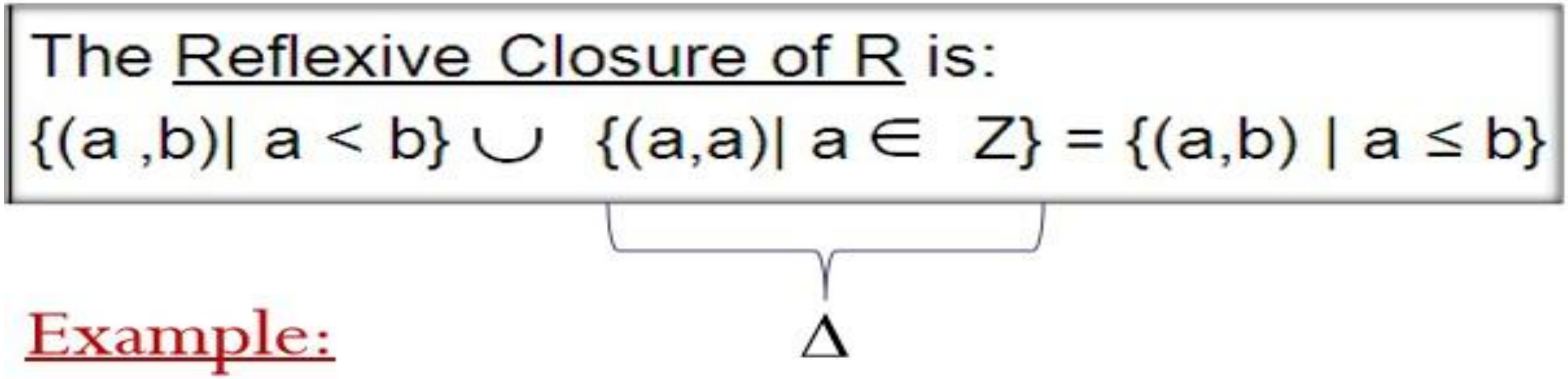
{(1,1),(1,2),(2,1),(3,2),**(2,2),(3,3)**}

Example 2

What is the reflexive closure of the relation:

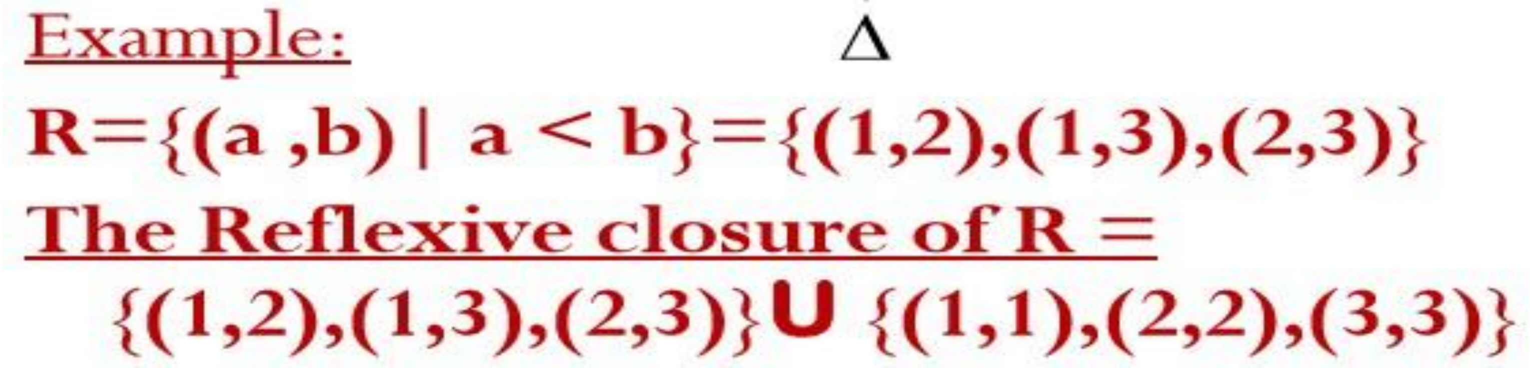
R={(a ,b) | a < b} on the set of integers?

Answer: R U Δ



The Reflexive Closure of R is:

$$\{(a,b) \mid a < b\} \cup \underbrace{\{(a,a) \mid a \in Z\}}_{\Delta} = \{(a,b) \mid a \leq b\}$$



Example:

R={(a ,b) | a < b}={(1,2),(1,3),(2,3)}

The Reflexive closure of R =

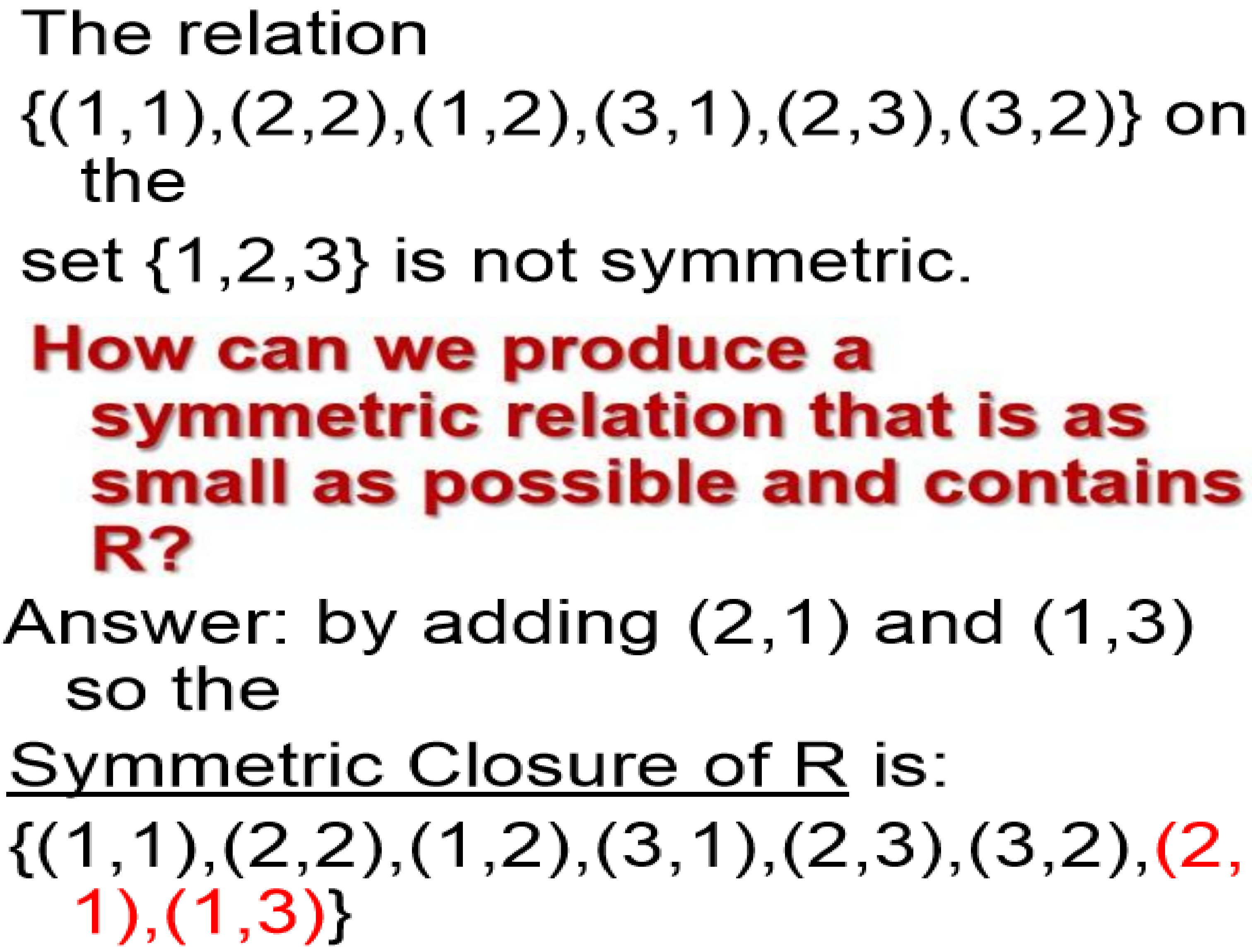
{(1,2),(1,3),(2,3)}U {(1,1),(2,2),(3,3)}

2) Symmetric closure

The ***symmetric closure*** of R is obtained by adding (b,a) to $\boldsymbol{R}$ for each (a,b) in $\boldsymbol{R}$.

i.e., it is $\boldsymbol{R} \cup \boldsymbol{R}^{-1}$

Example 1



The relation {(1,1),(2,2),(1,2),(3,1),(2,3),(3,2)} on the set {1,2,3} is not symmetric.

How can we produce a symmetric relation that is as small as possible and contains R?

Answer: by adding (2,1) and (1,3) so the

Symmetric Closure of R is:

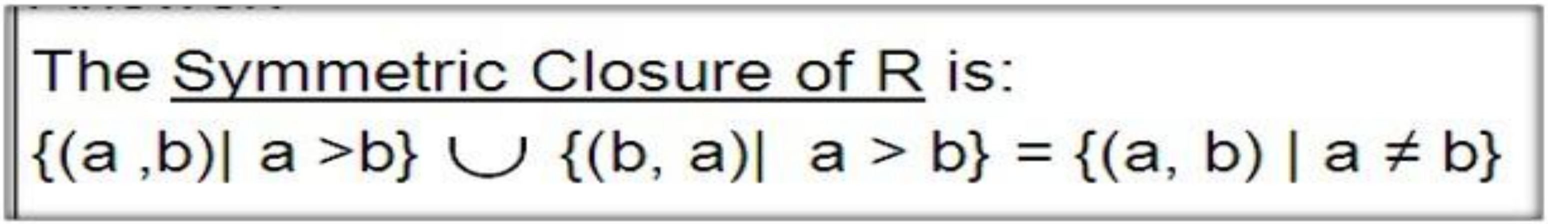
{(1,1),(2,2),(1,2),(3,1),(2,3),(3,2),(2,1),(1,3)}

Example 2

What is the **symmetric closure** of the relation:

$R=\{(a,b) \mid a > b\}$ on the set of positive integers?

Answer:



The Symmetric Closure of R is:

$\{(a,b) \mid a > b\} \cup \{(b,a) \mid a > b\} = \{(a,b) \mid a \neq b\}$

Example:

$$R = \{(3,1),(3,2),(2,1)\}$$

$$R^{-1} = \{(1,3),(2,3),(1,2)\}$$

$$R \cup R^{-1} = \{(3,1),(3,2),(2,1),(1,3),(2,3),(1,2)\}\}$$

3) Transitive closure

- The ***transitive closure*** or ***connectivity relation*** of R is obtained by repeatedly adding (a, c) to R for each (a, b), (b, c) in R.
  - *i.e.*, it is $$R^* = \bigcup_{n \in \mathbf{Z}^+} R^n$$

– **Or in term of zero-one matrices:**

– $M_R{}^* = M_R \vee M_{R[2]} \ldots \vee \ldots \ldots \vee M_{R[n]}$

Example 1

$R^{*} = R \cup R^{2} \cup R^{3}$

Ex: R={(1,1), (1,2), (2,1),(3,2)} on the set

A= {1, 2, 3}

- $\mathbf{R}^{*} = R \cup R^{2} \cup R^{3}$

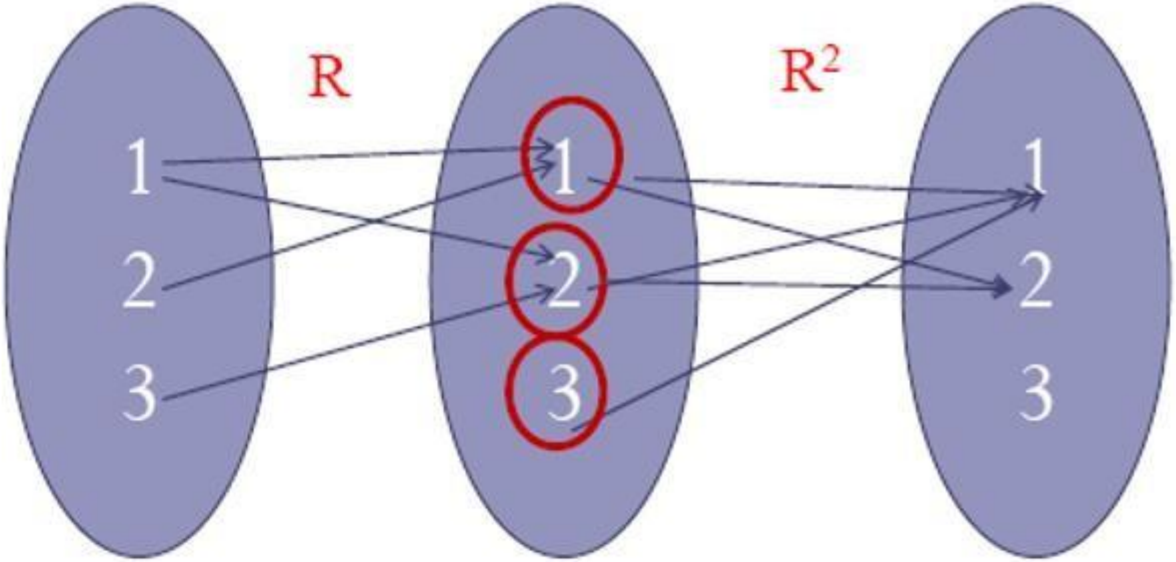
Example 1

$R^* = R \cup R^2 \cup R^3$

Ex: R={(1,1), (1,2), (2,1),(3,2)} on the set

A= {1, 2, 3}

$R^2 = R \circ R = \{(1,1),(1,2).(2,1),(2,2),(3,1)\}$



$R^3 = R^2 \circ R$

$= \{(1.1).(1.2).(2.1).(2.2),(3,1),(3,2)\}$

$R^* = R \cup R^2 \cup R^3$

$= \{(1,1),(1,2).(2,1),(2,2),(3,1),(3,2)\}$

- Or in term of zero-one matrices:
- $M_R^* = M_R \vee M_{R[2]} \ldots \vee \ldots \ldots \vee M_{R[n]}$

A= {1, 2, 3}

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^2} = M_R \Theta M_R$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^3} = M_{R^2} \Theta M_R$$

- **Or in term of zero-one matrices:**
- $\mathbf{M_R{}^{*}=M_R \vee M_{R[2]} \ldots \vee \ldots\ldots \vee M_{R[n]}}$

A= {1, 2, 3}

$$M_R = 0\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

R={(1,1), (1,2), (2,1),(3,2)}

$$M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$R^2 = R \circ R =$ {(1,1),(1,2).(2,1),(2,2),(3,1)}

$R^3 = R^2 \circ R =$ {(1,1),(1,2).(2,1),(2,2),(3,1),(3,2)}

$M_{R^*} =$ {(1,1),(1,2).(2,1),(2,2),(3,1),(3,2)}

Example 2

- Find M_{R*} for

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}} =$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Equivalence Relations

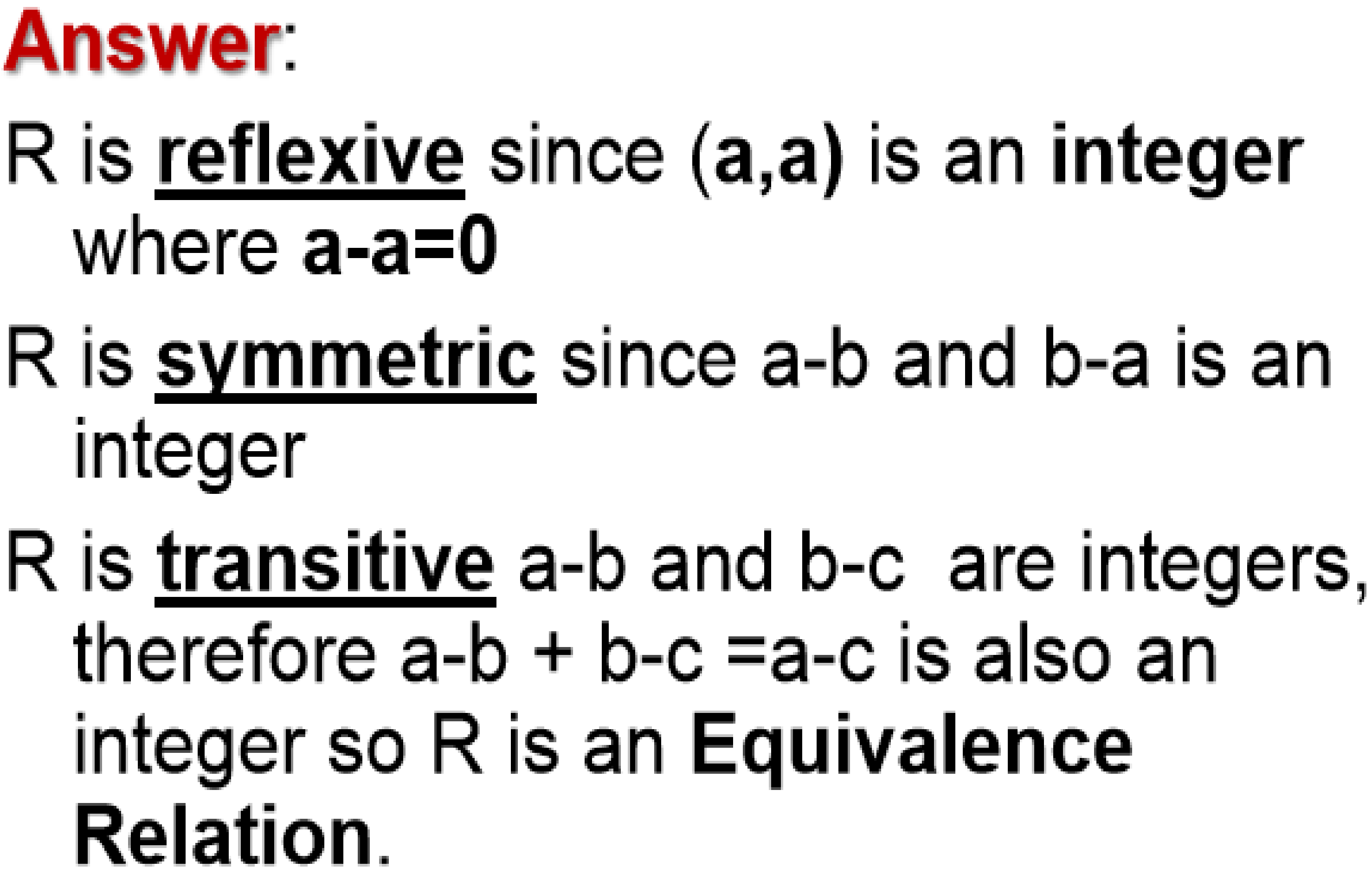
- An *equivalence relation* on a set *A* is simply any binary relation on *A* that is reflexive, symmetric, and transitive.

Example1:

A relation on the set of real numbers such that

R={(a, b) | a-b is an integer}

Equivalence Relations



Answer:

R is **reflexive** since (**a,a**) is an **integer** where **a-a=0**

R is **symmetric** since a-b and b-a is an integer

R is **transitive** a-b and b-c are integers, therefore a-b + b-c =a-c is also an integer so R is an **Equivalence Relation**.