Chapter 8

Advanced Counting Techniques

8.1 - 8.2

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
    - *Homogeneous Recurrence Relations*
    - *Nonhomogeneous Recurrence Relations*

8.1. Recurrence Relations (RR)

Recurrence Relations

In Chapter 5 we discussed how sequences can be defined recursively.
Recall that a recursive definition of a sequence specifies:

1. One or more initial terms and
2. A rule for determining subsequent terms from those that proceed them. A rule of the latter sort (whether or not it is part of a recursive definition) is called a **recurrence relation**. such relations can be used in studying and **solving counting problems.**

Recurrence Relations

- A *recurrence relation* (R.R., or just *recurrence*) for the sequence $\{a_n\}$ is ***an equation*** that expresses a_n in terms of **one** or **more** **previous terms** of the sequence $a_0, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer
  - *I.e.*, just a recursive definition, without the base cases.
- A **sequence** is called a **solution** of a ***recurrence relation*** if its **terms satisfy** the ***recurrence relation***.

Recurrence Relation Example

- Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \quad (n \geq 2).$$

- Which of the following are solutions?

$$a_n = 3n$$
$$a_n = 2^n$$
$$a_n = 5$$

Recurrence Relation Example

- Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \quad (n \geq 2).$$

- Which of the following are solutions?

$a_n = 3n$	Yes
$a_n = 2^n$	No
$a_n = 5$	Yes

Financial Application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\ P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Continued on next slide →

Financial Application

$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

$\vdots$

$P_n = (1.11)P_{n-1} = (1.11)^n\mathrm{P}_0 = (1.11)^n\, 10{,}000$

$P_n = (1.11)^n\, 10{,}000$

$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

8.2. SOLVING LINEAR RECURRENCE RELATIONS

K-LiHoReCoCo

General Solution Schemas

- A linear homogeneous recurrence of degree k with constant coefficients ("*k*-LiHoReCoCo") is a rec. rel. of the form

$$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k},$$

where the c_i are all real numbers, and $c_k \neq 0$.

- it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of *n*.
- it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
- the *degree is k* because a_n is expressed in terms of the previous *k* terms of the sequence.

Examples of Linear Homogeneous Recurrence Relations linear? Homogeneous? Which degree?

- $P_n = (1.11)P_{n-1}$
- $f_n = f_{n-1} + f_{n-2}$
- $a_n = a_{n-1} + a_{n-2}^2$
- $H_n = 2H_{n-1} + 1$
- $B_n = nB_{n-1}$

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.
- Algebraic manipulation yields the *characteristic equation*:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation (C.E).
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 ($r_1 \neq r_2$)

. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation is $r^2 - r - 2 = 0$.
Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and
only if $a_n = α_1 2^n + α_2(-1)^n$, for some constants $α_1$ and $α_2$.

To find the constants $α_1$ and $α_2$, note that

$a_0 = 2 = α_1 + α_2$ and $a_1 = 7 = α_1 2 + α_2(-1)$.

Solving these equations, we find that $α_1 = 3$ and $α_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

Using Theorem 1

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$
$$a_n = \quad a_{n-1} + 2a_{n-2}$$
$$r^2 - c_1 r - c_2 = 0$$
$$c_1 = 1,\ c_2 = 2$$
$$r^2 - r - 2 = 0$$

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$.
- Solution: Use theorem 1:
  - We have $c_1 = 1$, $c_2 = 2$
  - The characteristic equation is: $r^2 - r - 2 = 0$
  - Solve it:

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2} = 2 \text{ or } -1.$$

- so, $r = 2$ or $r = -1$.
- So, $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$.

(Using the quadratic formula here.)

$$ax^2 + bx + c = 0 \Leftrightarrow$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example Cont…

- To find α_1 and α_2, just solve the equations for **the initial conditions a_0 and a_1**:

$$a_0 = 2 = \alpha_1 2^0 + \alpha_2 (-1)^0$$

$$a_1 = 7 = \alpha_1 2^1 + \alpha_2 (-1)^1$$

Simplifying, we have the pair of equations:

$$2 = \alpha_1 + \alpha_2$$

$$7 = 2\alpha_1 - \alpha_2$$

which we can solve easily by substitution:

$$\alpha_2 = 2-\alpha_1; \quad 7 = 2\alpha_1 - (2-\alpha_1) = 3\alpha_1 - 2;$$

$$9 = 3\alpha_1; \quad \boldsymbol{\alpha_1 = 3}; \quad \boldsymbol{\alpha_2 = -1}.$$

- **Using α_1 and α_2** in $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$, *then* **our final answer is:**

$$a_n = 3\cdot 2^n - (-1)^n$$

Check: $\{a_{n\geq 0}\} = 2, 7, 11, 25, 47, 97 \ldots$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for all $n \geq 0$,

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$. The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = α_1 3^n + α_2 n(3)^n$$

where $α_1$ and $α_2$ are constants.

To find the constants $α_1$ and $α_2$, note that

$$a_0 = 1 = α_1 \quad \text{and} \quad a_1 = 6 = α_1 \cdot 3 + α_2 \cdot 3.$$

Solving, we find that $α_1 = 1$ and $α_2 = 1$.

Hence,

$$a_n = 3^n + n3^n .$$

More Examples

- $a_n = (1.02)a_{n-1}$

- $a_n = (1.02)a_{n-1}$
  linear
  constant coefficients
  homogeneous
  degree 1

- $a_n = (1.02)a_{n-1} + 2^{n-1}$

- $a_n = (1.02)a_{n-1} + 2^{n-1}$
  linear
  constant coefficients
  nonhomogeneous
  degree 1

- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$
  linear
  constant coefficients
  nonhomogeneous
  degree 3

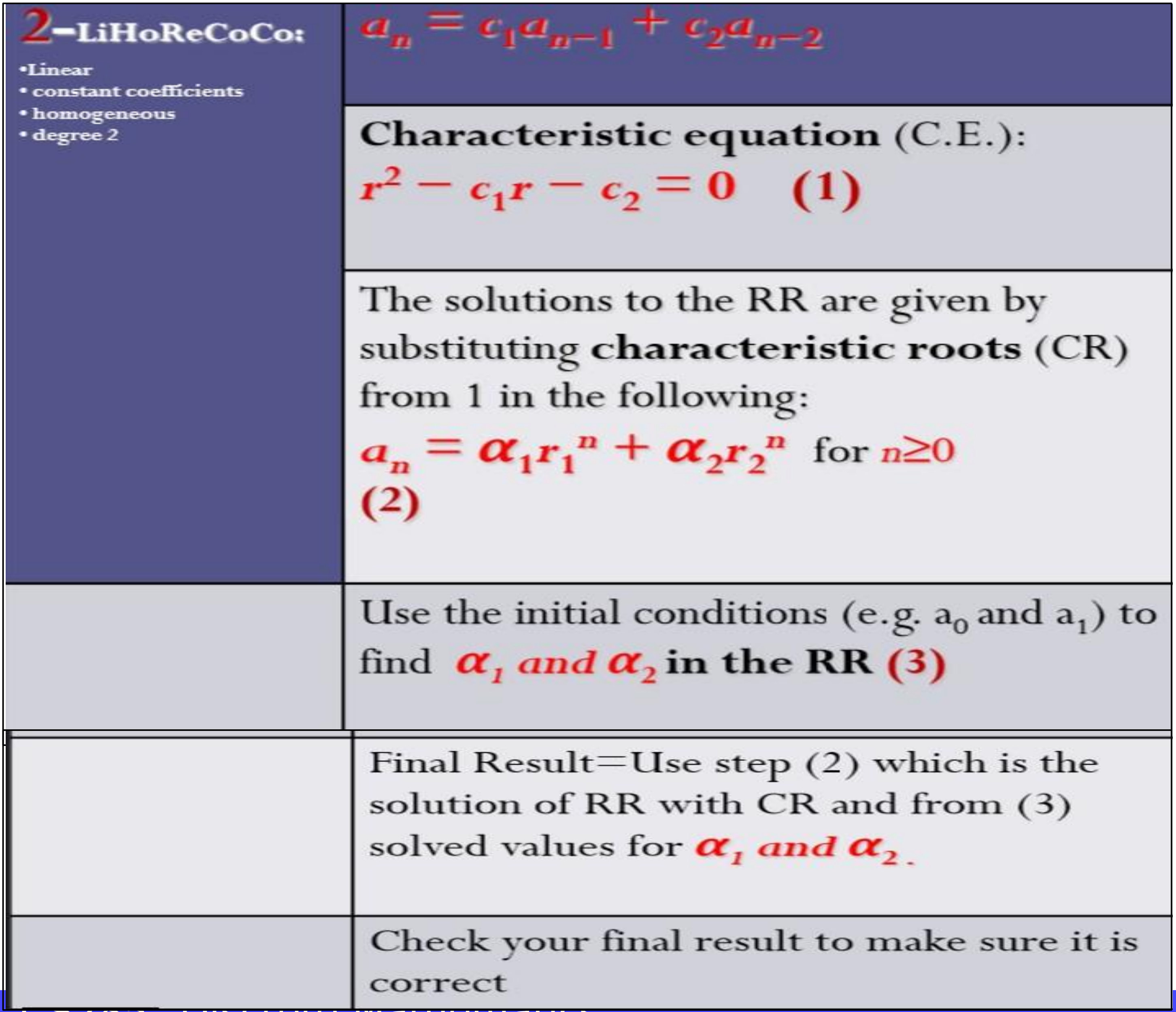
- $a_n = ca_{n/m} + b$

- $a_n = ca_{n/m} + b$
  does not have the right form

- $a_n = na_{n-1} + n^2a_{n-2} + a_{n-1}a_{n-2}$

- $a_n = na_{n-1} + n^2a_{n-2} + a_{n-1}a_{n-2}$
  nonlinear
  coefficients are not constants
  homogeneous
  degree 2

Solution Procedure: 2-LiHoReCoCo



2-LiHoReCoCo: •Linear • constant coefficients • homogeneous • degree 2	$a_n = c_1 a_{n-1} + c_2 a_{n-2}$
	Characteristic equation (C.E.): $r^2 - c_1 r - c_2 = 0$ (1)
	The solutions to the RR are given by substituting **characteristic roots** (CR) from 1 in the following: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \geq 0$ (2)
	Use the initial conditions (e.g. a_0 and a_1) to find α_1 *and* α_2 **in the RR** (3)
	Final Result=Use step (2) which is the solution of RR with CR and from (3) solved values for α_1 *and* α_2.
	Check your final result to make sure it is correct

Linear
Homogeneous
Recurrence of
degree 2 with
Constant
Coefficient

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^m + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

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Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k **distinct** roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots. + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Using Theorem 3

Find the solution to the RR $a_n = 6a_{n-1} - 11 a_{n-2} + 6 a_{n-3}$
with the initial conditions $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

The General Case with Repeated Roots Allowed

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

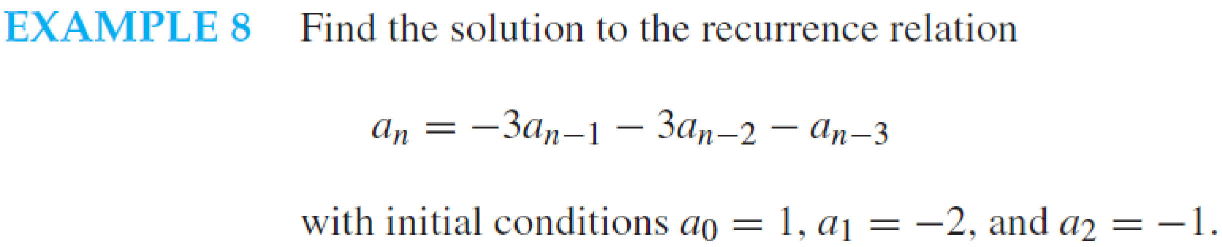
Using Theorem 4

Example 7: Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one).
What is the form of the general solution?

Solution:

By Theorem 4, the general form of the solution is

Using Theorem 4

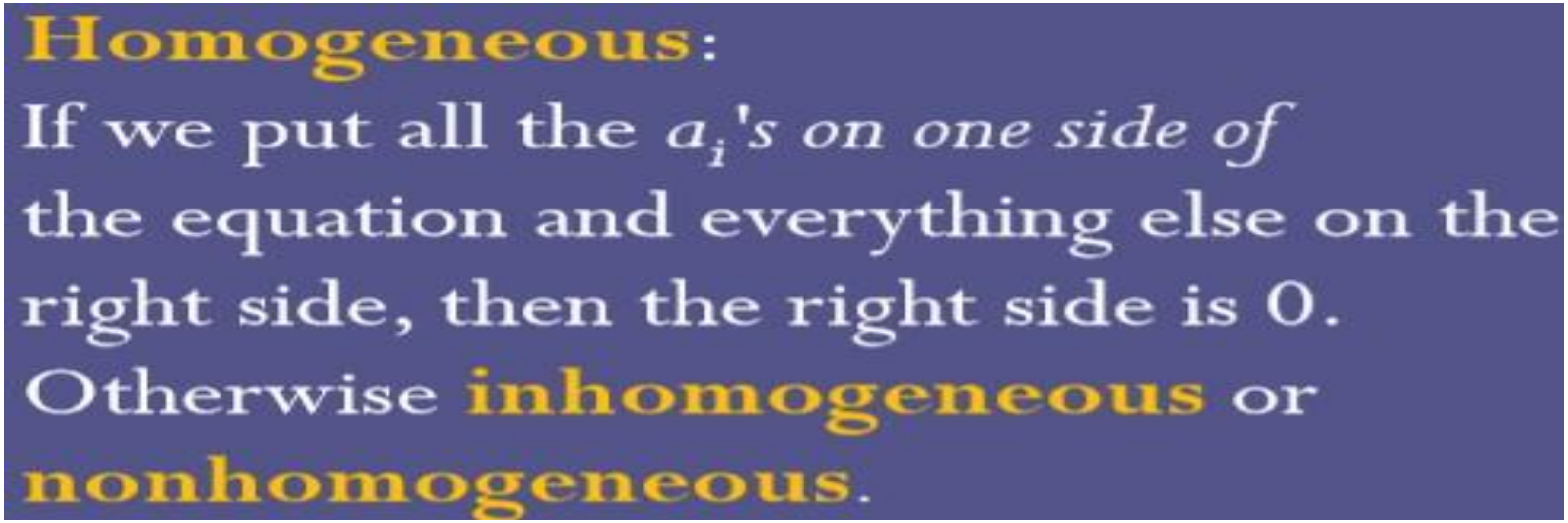


EXAMPLE 8 Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

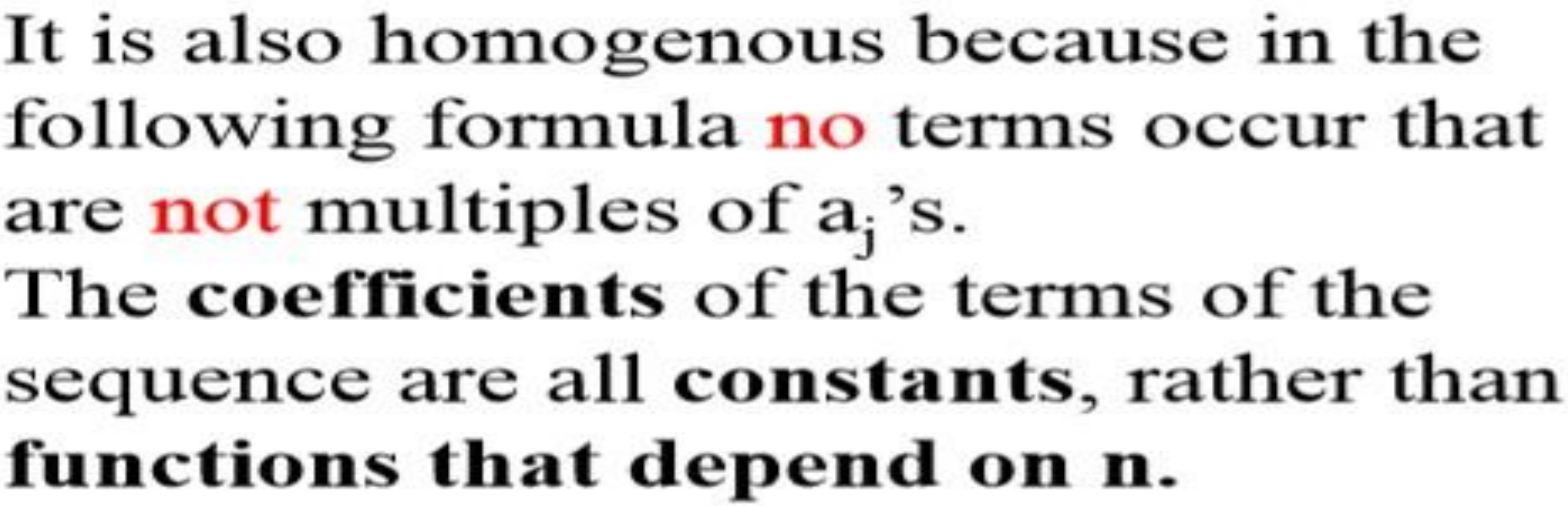
Homogeneous & Nonhomogenes



Homogeneous:

If we put all the *a_i's on one side of* the equation and everything else on the right side, then the right side is 0.

Otherwise **inhomogeneous** or **nonhomogeneous**.



It is also homogenous because in the following formula **no** terms occur that are **not** multiples of a_j's.

The **coefficients** of the terms of the sequence are all **constants**, rather than **functions that depend on n.**

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (LiNoReCoCos)

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k} + F(n)\ ,$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k}\ ,$$

is called the associated homogeneous recurrence relation.

LiNoReCoCos – cont.

- Linear *nonhomogeneous* RRs with constant coefficients may (unlike Li**Ho**ReCoCos) contain some terms $F(n)$ that depend *only* on n (and *not* on any a_i's). General form:

$$\underbrace{a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}}_{\text{The \textit{associated homogeneous recurrence relation} (associated LiHoReCoCo).}} + F(n)$$

LiNoReCoCos – cont.

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

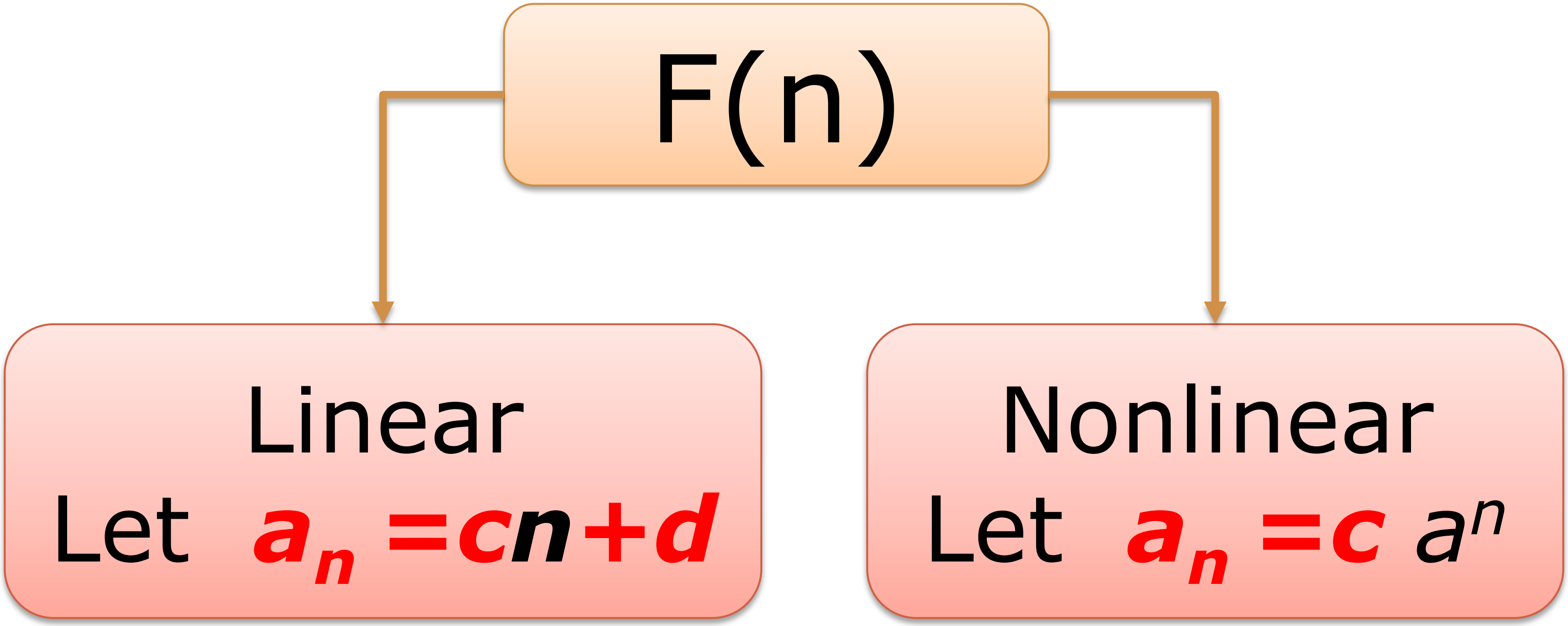
$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Solution= $a_n = \{a_n^{(p)} + a_n^{(h)}\}$

F(n) Types



F(n)

Linear

Let $a_n = cn + d$

Nonlinear

Let $a_n = c\,a^n$

When F(n) is linear

Linear
Let **$a_n = cn + d$**

Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

When F(n) is nonlinear

Nonlinear
Let **$a_n = c$** a^n

Find all solutions of the recurrence relation
$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.