Counting

Chapter 6.1 - 6.3 - 6.4

Chapter Summary

- The Basics of Counting
- Permutations and Combinations
- Binomial Coefficients and Identities

The Basics of Counting

Section 6.1

Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples

The Basics of Counting

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there? The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this section.

Counting problems arise throughout mathematics and computer science. For example, we must count the successful outcomes of experiments and all the possible outcomes of these experiments to determine probabilities of discrete events. We need to count the number of operations used by an algorithm to study its time complexity.

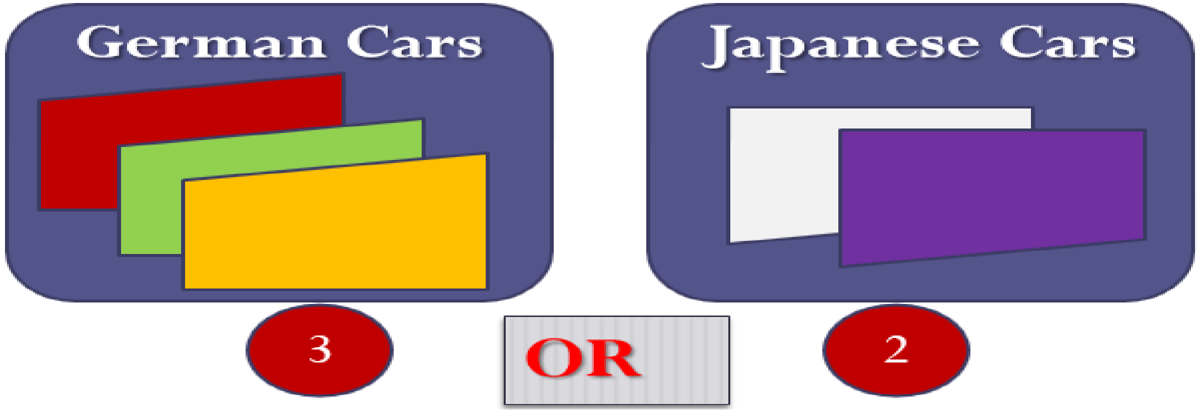
We will introduce the basic techniques of counting in this section. These methods serve as the foundation for almost all counting techniques.

Reminder of Few Definitions Concerning SETS

- A **set** is an **unordered collection of distinct objects**, which we call the **elements of the set.**
- The set of **no elements** is called the **empty set**.
- If A is a finite set, **|A|** denotes the number of elements in A, which is called **the cardinality of A.**
- The **union** of sets **A and B**, denoted **A** ∪ **B**, **is the set of all elements in A or B.**
- The **intersection** of sets **A and B**, denoted **A** ∩ **B**, is the **set of all elements in both A and B**.

Warm-Up Questions

- Suppose you want to buy a car and there are 2 rooms in the gallery. In one room you have 3 different German cars and in the other room there are 2 different Japanese cars.
- **What is the number of choices to buy a car?**



Sum Rule = n_1+n_2 = 3+ 2 =5

Independent

Independent because choosing a German car is independent of choosing a Japanese car, meaning if you choose to buy a German car prevents you from buying a Japanese car.

Basic Counting Principles

The SUM Rule:

- ***The sum rule:*** If a first task can be done in n_1 ways and a second task in **n2** ways, and **if only one of these tasks can be done but not both**, then there are **n1 + n2** ways to do either task.
- **Phrased in terms of sets:** If a first task is to choose **an element** from **set S** and a second task is to choose **an element** from **set T**, and **S** and **T** are two disjoint finite sets, then the number of ways to choose an element from either set **is** $|S \cup T| = |S| + |T|$.

THE RULE OF SUM
If A and B are disjoint sets then $|A \cup B| = |A| + |B|$

A count decomposes into a set of **independent** counts
("elements of counts are **alternatives**")

The Rule of *SUM*

Example

- Suppose statement labels in a programming language must be **a single letter** <u>**or**</u> a **single decimal digit**.

What is the number of the possible labels?

Since **a label cannot be both at the same time**, the number of labels = the number of letters + the number of decimal digits

= 26 + 10 = 36.

Warm- Up Questions

- Suppose you should travel from country **A** to Country **C** and there is a transit in the middle in country **B**.
- Moreover suppose that you can reach point B from A in **3** different ways and point C from B in **2** different ways.
- **How many ways are there to reach C from A?**

Product Rule = $n_1 * n_2$ = 3* 2 =6

Dependent

Dependent because C depends on B and B depends on A

Basic Counting Principles-The PRODUCT Rule:

- **The product rule:** Suppose that a procedure can be broken down into two tasks. If there are **n1** ways to do the first task and **n2** ways to do the second after the first task has been done, then there are **n1n2** ways to complete the procedure.
- **Phrased in terms of sets:** If a task to choose **an element** from **set S** is followed by a second task to choose **an element** from set **T**, where both **S** and **T** are **finite sets,** then the number of ways to choose the pair of elements is
  $|S \times T| = |S| \cdot |T|$.

THE RULE OF PRODUCT

$$|A \times B| = |A|\,|B|$$

Product Example

- How many phone line could be installed in Riyadh city, if the number of phone should be composed of **7 numbers** starting with the number **4 or 2,**

1. In case repetition is allowed
2. And then if repetition is not allowed .

Solution: In case repetition is allowed :

1	2	3	4	5	6	7
2	10	10	10	10	10	10
2 or 4	0 - 9	0 - 9	0 - 9	0 - 9	0 - 9	0 - 9

$(2)(10)\ (10)\ (10)\ (10)\ (10)\ (10)=(2)(10)^6=2{,}000{,}000$ phone line

Solution: In case repetition is <u>not</u> allowed :

1	2	3	4	5	6	7
2 choices	9 choices	8 choices	7 choices	6 choices	5 choices	4 choices
2 or 4	0 - 9	0 - 9	0 - 9	0 - 9	0 - 9	0 - 9

$(2)(9)\ (8)\ (7)\ (6)\ (5)\ (4)$
$=(2)(60480)=120{,}960$ phone line

- Riyadh car plates are of the form 3 letters and 3 digits.
- How many plates can be produced?
- L1L2L3D1D2D3
  - where the L_i are letters in {A…Z}
  - and the D_i are digits in {0…9}

L1	L2	L3	D1	D2	D3
26	26	26	10	10	10

$26^3 * 10^3$ plates

Examples

- The entry code to our office is 4 characters long where each character is a number between $\{0 \ldots 9\}$.

  e.g., 1012, 2561.

  How many door codes are there?

- A computer system's password must be 6 characters long, where each character is in $\{a \ldots z, A \ldots Z\}$.

  e.g., $abcdEF$, $DHrsAQ$, $Mickey$.

  How many passwords are there?

- Hong Kong car plates are of the form $L_1L_2D_1D_2D_3D_4$ where the L_i are letters in $\{A \ldots Z\}$ and the D_i are digits in $\{0, \ldots 9\}$.

  e.g., $AB1234$, $EC1357$.

  How many car plates are there?

The entry code to our office is 4 characters long where each character is a number between $\{0 \dots 9\}$.
E.G., 1012, 2561.
How many door codes are there?

A door code is of the form $X_1 X_2 X_3 X_4$ where,
for $i = 1, 2, 3, 4$, $X_i \in \{0, 1, \dots, 9\}$
($\in$ means *is in* or *is a member of*)

There are
10 possibilities for X_1 10 possibilities for X_2
10 possibilities for X_3 10 possibilities for X_4

$\Rightarrow 10 \times 10 \times 10 \times 10 = 10^4 = 10,000$ possible door codes.

A computer system's password must be 6 characters long, where each character is in $\{a \ldots z, A \ldots Z\}$.
E.G., $abcdEF$, $DHrsAQ$, $Mickey$.
How many passwords are there?

A password is of the form $D_1D_2D_3D_4D_5D_6$ where
for $i = 1, 2, 3, 4, 5, 6, \quad D_i \in \{a \ldots z, A \ldots Z\}$.
Each D_i has 52 possible choices so,
the total number of passwords is

$$52 \times 52 \times 52 \times 52 \times 52 \times 52 = 52^6$$

Hong Kong car plates are of the form $L_1L_2D_1D_2D_3D_4$
where the L_i are letters in $\{A \ldots Z\}$
and the D_i are digits in $\{0, \ldots 9\}$.

e.g., $AB1234$, $EC1357$.
How many car plates are there?

Each L_i has 26 possibilities and
each D_i has 10 possibilities
so the total number of car plates is

$$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^4 = 6,760,000$$

Example

- Statement labels in Basic can be either

- a single letter **or**
- a letter followed by a digit.

Find the number of possible labels.

We can partition the **set of all labels** *L into the* **disjoint subsets** consisting of

- the set of single letter labels ***S*** and
- the set of single letters followed by a digit ***D*** and

$$L = S \cup D.$$

- Use the rule of sum to compute the cardinality of ***L*** *if* we can compute the cardinality of ***D.***
- The elements of D are ordered pairs of the form **[a,d]** *where* ***a*** *is an* ***alphabetic character*** *and* ***d*** *is a* ***digit***.
- By the rule of product the cardinality of ***D*** is the product of the cardinality of the two sets:
  - ***(the alphabetic characters)(the decimal digits)*** = **(26)(10)= 260**.

The cardinality of *L* *is 26 + 260 = 286*.

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion - Exclusion

The principle of Inclusion-Exclusion generalized the sum rule the case of **non-empty intersection**:

INCLUSION-EXCLUSION: If A and B are sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This says that when counting all the elements in A or B, **if we just add the sets**, **we have double-counted the intersection**, and **must therefore subtract it out**.

THE PRINCIPLE OF INCLUSION-EXCLUSION

If A and B are not disjoint:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Don't count objects in the intersection of two sets more than once!

Visualize.

Inclusion - Exclusion

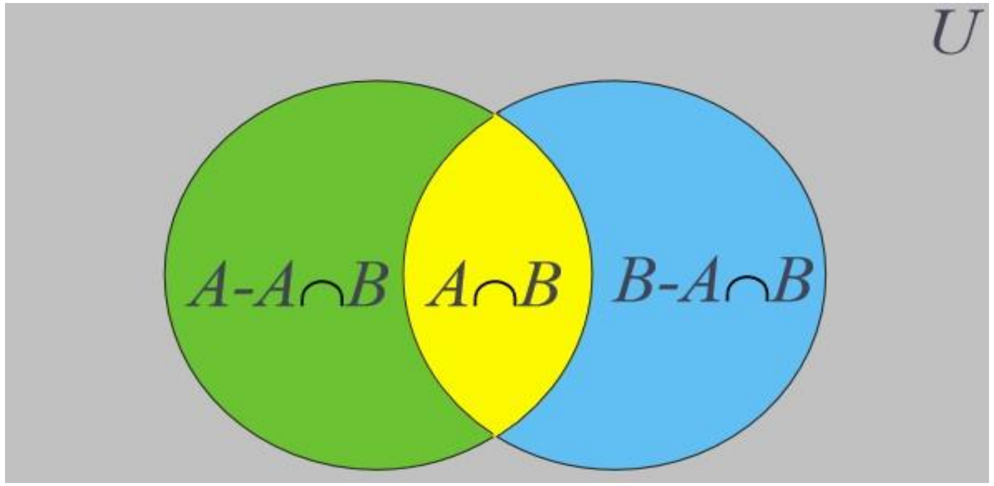


Diagram gives proof Inclusion-Exclusion principle:

Green Yellow Blue

$$|A\cup B| = |A-A\cap B|+|A\cap B|+|B-A\cap B|$$

$$=(|A-A\cap B|+|A\cap B|)+(|B-A\cap B|+|A\cap B|)-|A\cap B|$$

$$=|A|+|B|-|A\cap B|$$

Example

A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these people majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution

Solution: To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants. Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. By the principle of inclusion–exclusion, the number of students who majored either in computer science or in business (or both) equals

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$$

We conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in business.

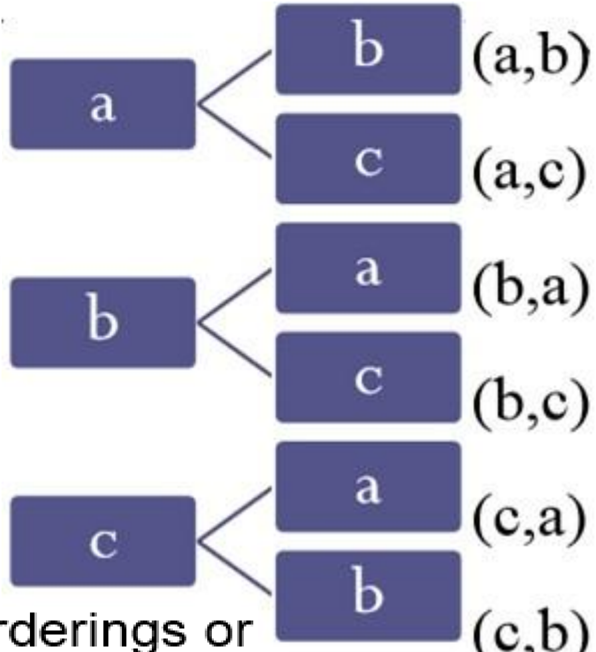
Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations



- Suppose we have 3 letters a, b, c
- We want to select 2 letters at a time.
- Then,

Permutation

{ ab, ac, ba, bc, ca,cb }

- A *permutation is an arrangement.*
- **Order matters.**
- After selecting the objects, two different orderings or arrangements constitute different permutations.
- **A *permutation* is an ordered arrangement of objects.**
- The number of permutations of r distinct objects chosen from n distinct objects is denoted P(n,r).
- P(3,2) = 3! / (3-2)! = 3! =3*2*1=6

P(n,r) = n! / (n-r)!

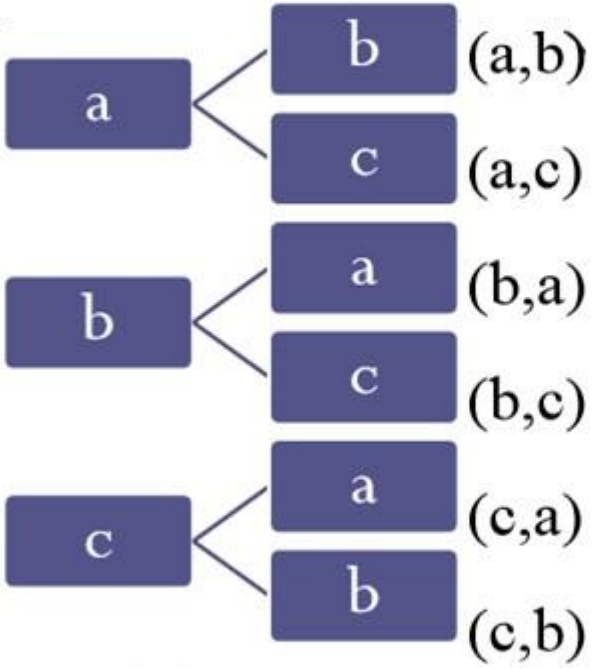
Cominations

- A combination is an unordered selection of elements from some set.
- The number of combinations of r distinct objects chosen from n distinct objects is denoted by C(n,r) or nCr or $\binom{n}{r}$, and is read "n choose r."
- C(n,r) = P(n,r)/r! = n!/((n-r)!r!)

The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section* 6.4.)

Combinations

- Suppose we have 3 letters a, b, c
- We want to select 2 letters at a time.
- Then,



Combinations

{ ab, ac, bc}

- **Order doesn't matter.**

C(3,2) = 3!/((3-2)!2!) = 3*2/2 = 3

C(n,r) = P(n,r)/r! = n!/((n-r)!r!)

Permutation, Combination, Factorial

Formulas for Discrete Math Counting

1) Formula for Permutation:

$$P(n,r) = \frac{n!}{(n-r)!}$$

2) Formula for Combination:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

3) Formula for Factorial:

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1.$$

Example

How many ways 3 birds can be taken from among 8 birds if order matters?

Solution

Permutation: $P(n,r) = \frac{n!}{(n-r)!}$

Here, $n = 8$, $r = 3$

$$P(8,3) = \frac{8!}{(8-3)!}$$

$$= \frac{8 \times 7 \times 6 \times 5!}{5!}$$

After simplify this, we get

$$= 336$$

In 336 ways 3 birds can be taken from among 8 birds.

Example

How many ways 2 books can be chosen from among 10 books?

Solution

Combination: $C(n,r) = \frac{n!}{r!(n-r)!}$,

Here n = 10, r =2

$$C(10, 2) = \frac{10!}{2!(10-2)!}$$

$$= \frac{10 \times 9 \times 8!}{8! \times 2 \times 1}$$

After simplify this, we get

= 45

In 45 ways 2 guides can be chosen from among 10 guides.

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

Solution

note that the runners are distinct and that the medals are ordered.
The solution is $P(8,3) = 8! / (8-3)! = 336$.

We need to create a team of 5 player for the competition out of 10 team members. How many different teams is it possible to create?

- When creating a team **we do not care about the order** in which players were picked. It is important that the player is in. Because of that we need to consider **unordered sets of people.**
- C(10,5) = 10!/(10-5)!5! = (10.9.8.7.6) / (5 4 3 2 1) = 2.3.2.7.3= 6.14.3= 6.42= **252**

Binomial Coefficients and Identities

Section 6.4

Section Summary

- The Binomial Theorem
- Pascal’s Identity and Triangle

Powers of Binomial Expressions

- **Definition**: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)
- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.

The number of ways of choosing a k-element subset from a set of size n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This term is called a binomial coefficient.
Be aware that there are other, alternative, notations for the same thing, occasionally used, e.g., $C(n,k)$ and nCk

binomial is a polynomial that is the sum of **two** terms (x+y)

Polynomial is an expression consisting of variables and coefficients

Binomial Theorem

Binomial Theorem: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

The Binomial Theorem

$$(x+y) = \binom{1}{0}x + \binom{1}{1}y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}x^1y^1 + \binom{2}{2}y^2$$

$$\begin{aligned}(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3\end{aligned}$$

For any integer $n \geq 0$, **Binomial Theorem**

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

or, in summation notation,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

1. **What is $(x+1)^4$?**
2. **What is $(2+y)^4$?**
3. **What is $(x+y)^4$?**

Solution

What is $(x+1)^4$?

By applying the binomial theorem

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$$

What is $(2+y)^4$?

$$(2+y)^4 = 16 + 32y + 24y^2 + 8y^3 + y^4.$$

What is $(x+y)^4$?

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Pascal Triangle

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\dots$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

	powers
1	
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5
1 6 15 20 15 6 1	6
1 7 21 35 35 21 7 1	7
1 8 28 56 70 56 28 8 1	
…	

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: First, note that this expression equals $(2x + (-3y))^{25}$. By the binomial theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!\,12!} 2^{12} 3^{13}.$$