Mathematical Induction
5.1 & 5.3

Chapter Summary

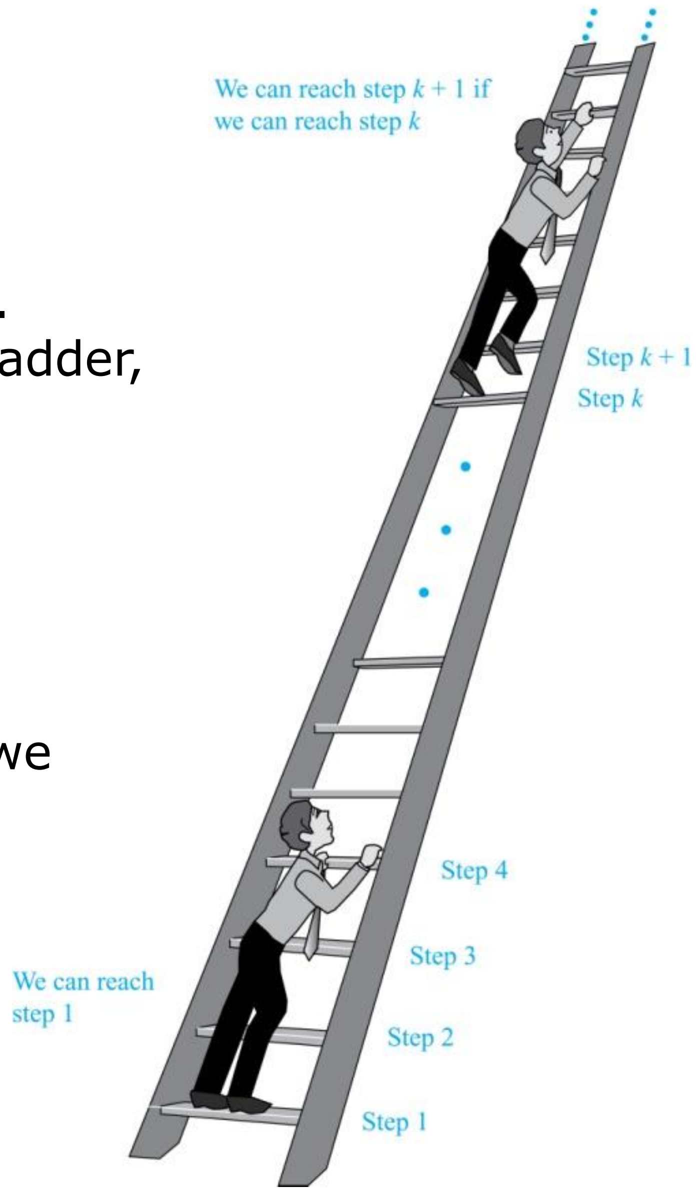
1. Mathematical Induction
2. Examples of Proof by Mathematical Induction
3. Recursive Definitions

Climbing an Infinite Ladder

Suppose we have an infinite ladder:
1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

The principle of Mathematic induction:
1. Show we can reach the first step
2. Show that if we can reach step k, then we can reach step k+1

if we establish these 2 things, we know we can reach all the steps of the ladder.



This example motivates proof by mathematical induction.

Principle of Mathematical Induction (MI)

Principle of Mathematical Induction: To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete these steps:

1. **Basis Step:** Show that P(1) is true.
2. **Inductive Step:** Show that $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Mathematical induction can be expressed as the rule of inference

$$(P(1) \wedge \forall k\, (P(k) \rightarrow P(k + 1))) \rightarrow \forall n\, P(n),$$

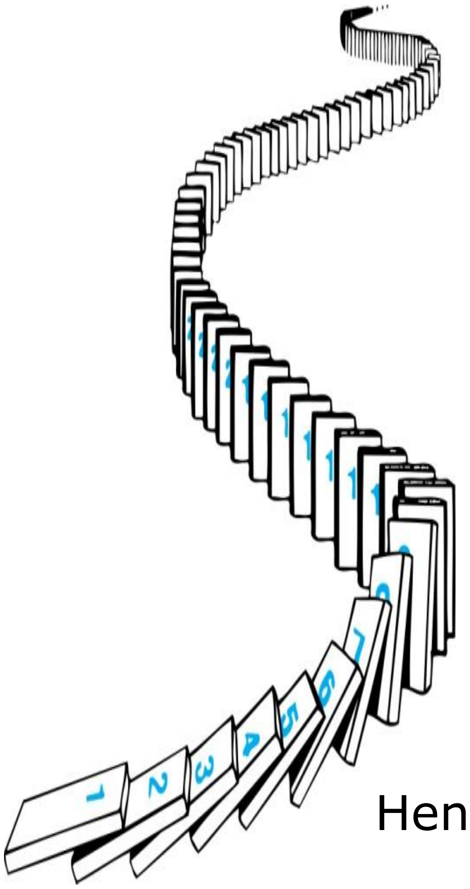
where the domain is the set of positive integers.

Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point *b* where *b* is an integer. We will see examples of this soon.

Remembering How Mathematical Induction Works - Domino Effect

Consider an infinite sequence of dominoes, labeled $1,2,3, \ldots,$ where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

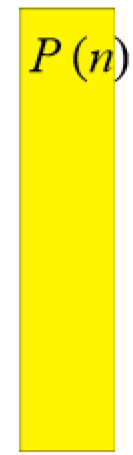
Mathematical Induction

Suppose we have a sequence of propositions which we would like to prove:

$P(0), P(1), P(2), P(3), P(4), \ldots P(n), \ldots$

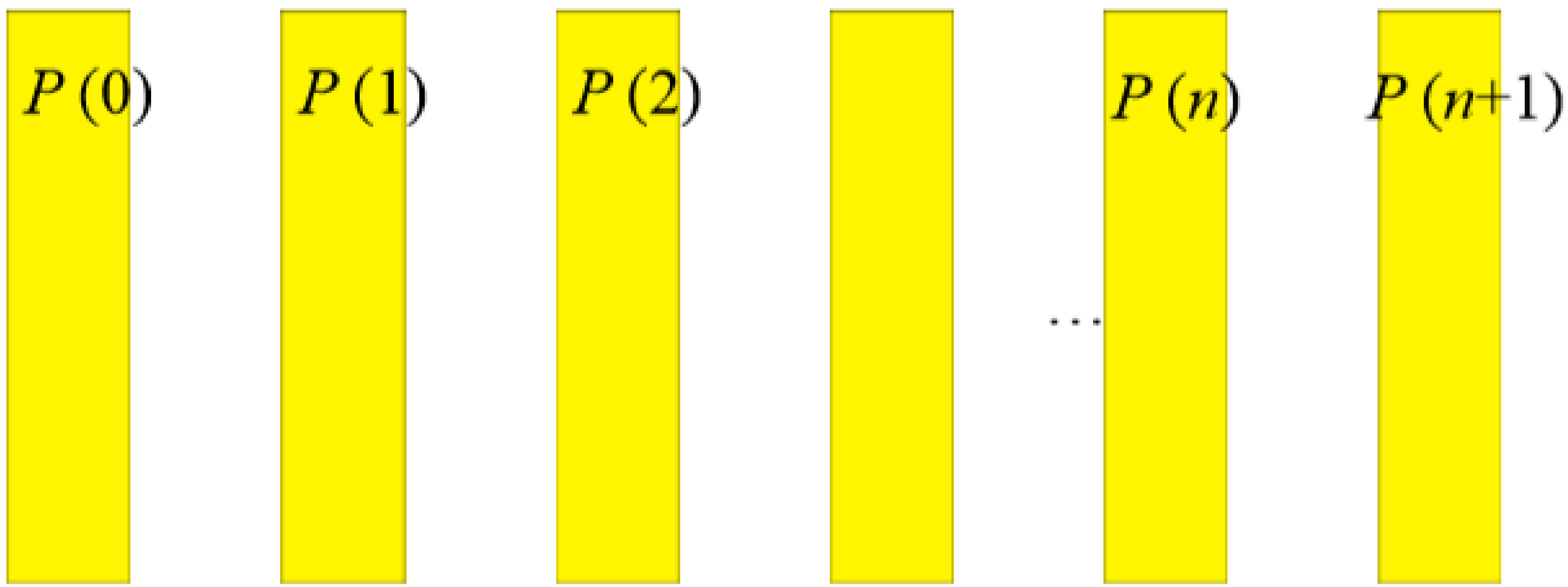
EG: $P(n) =$

We can picture each proposition as a domino:



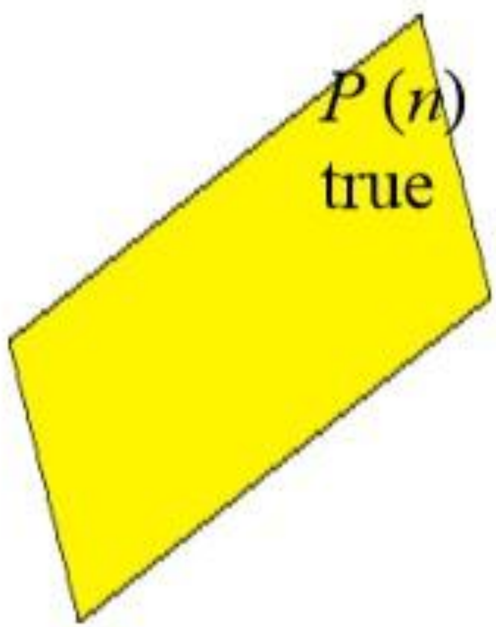
Mathematical Induction

So sequence of propositions is a sequence of dominos.



Mathematical Induction

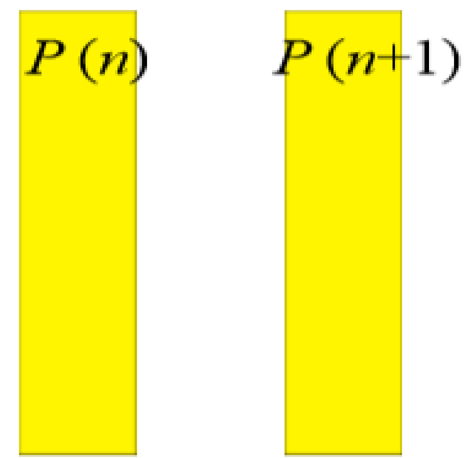
When the domino falls (to right), the corresponding proposition is considered true:



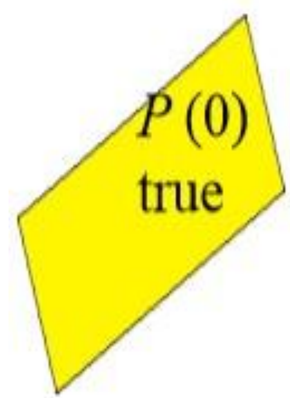
Mathematical Induction

Suppose that the dominos satisfy two constraints.

1) Well-positioned: If any domino falls (to right), next domino (to right) must fall also.

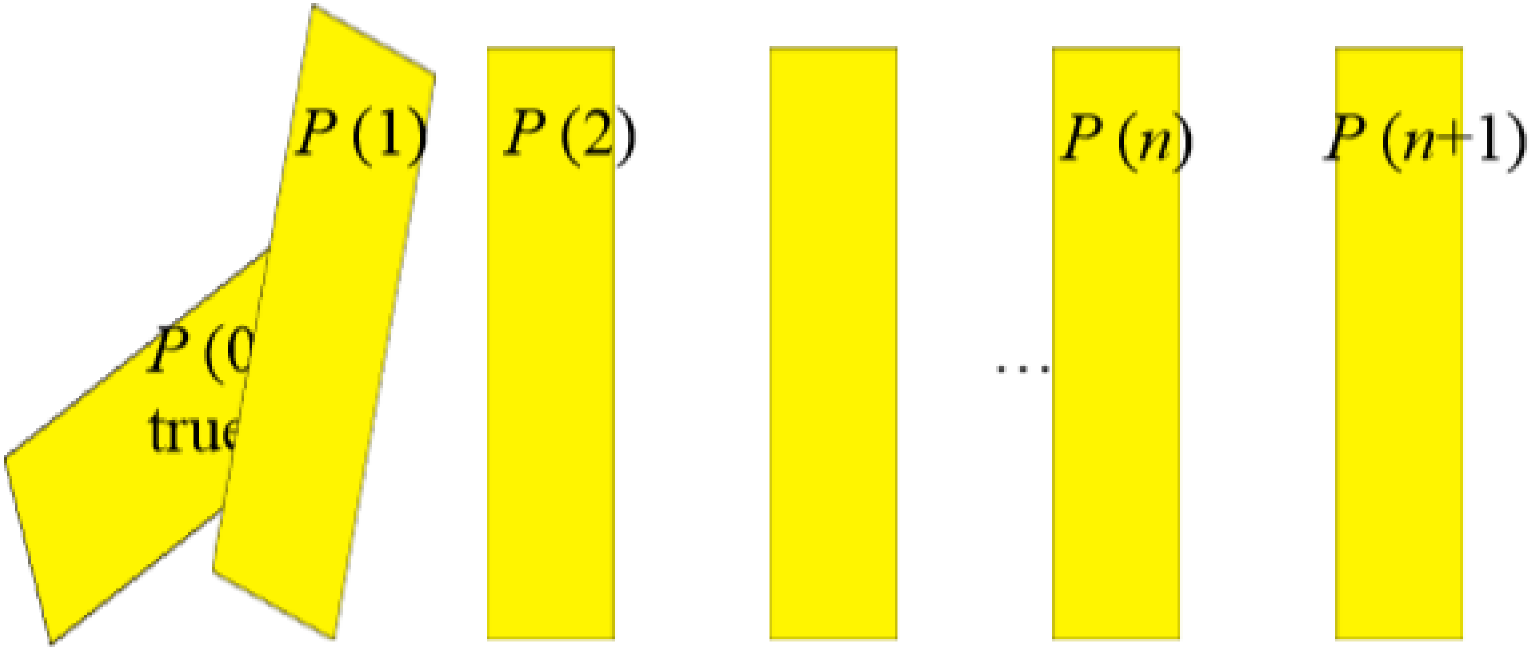


2) First domino has fallen to right



Mathematical Induction

Then we can conclude that all the dominos fall!



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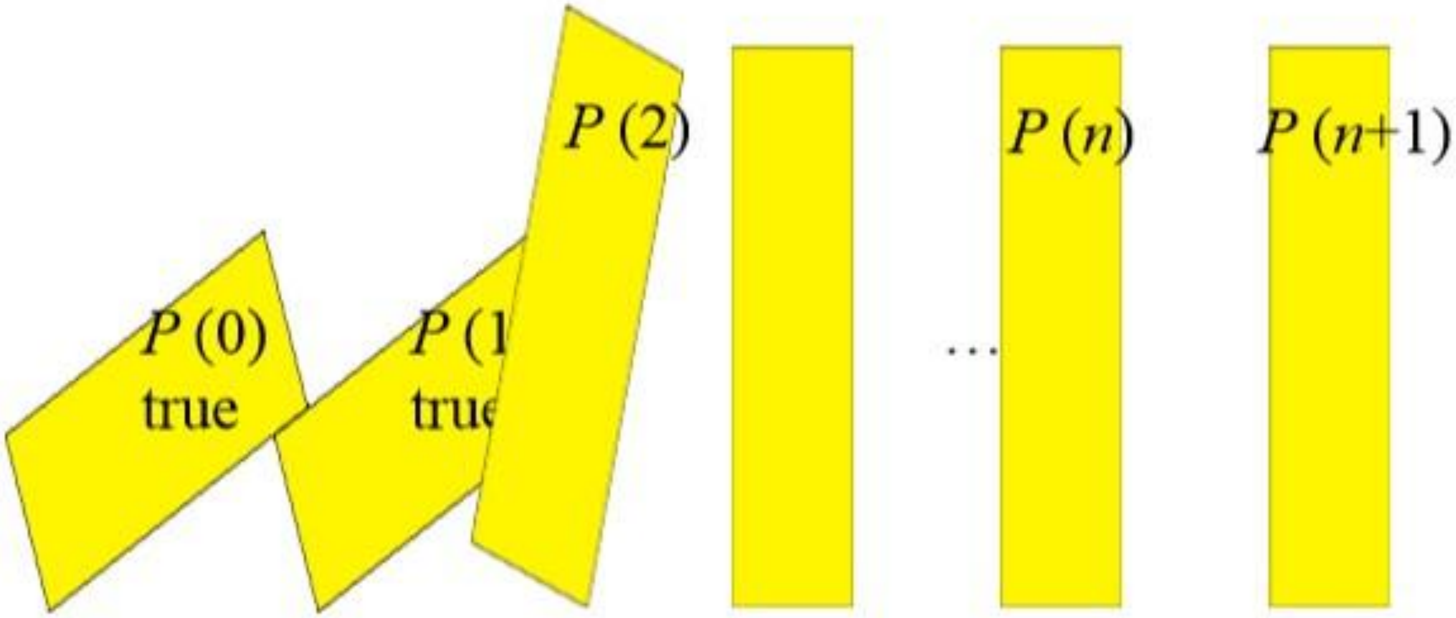

Mathematical Induction

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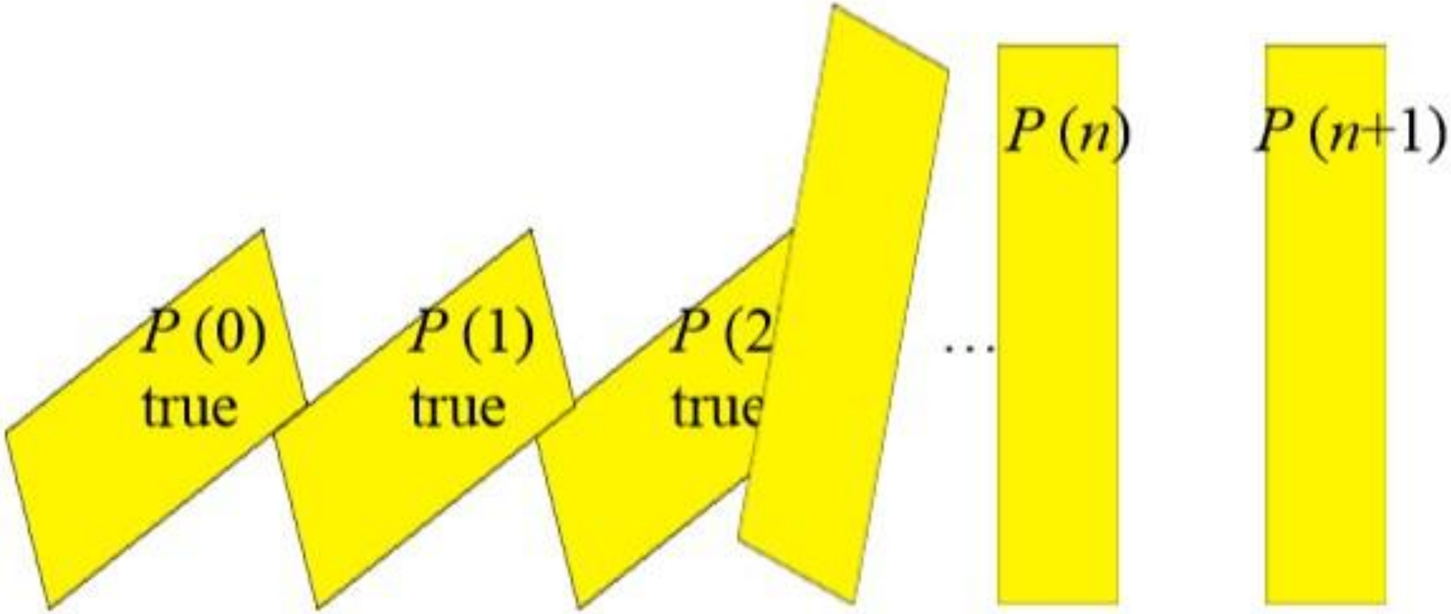

Mathematical Induction

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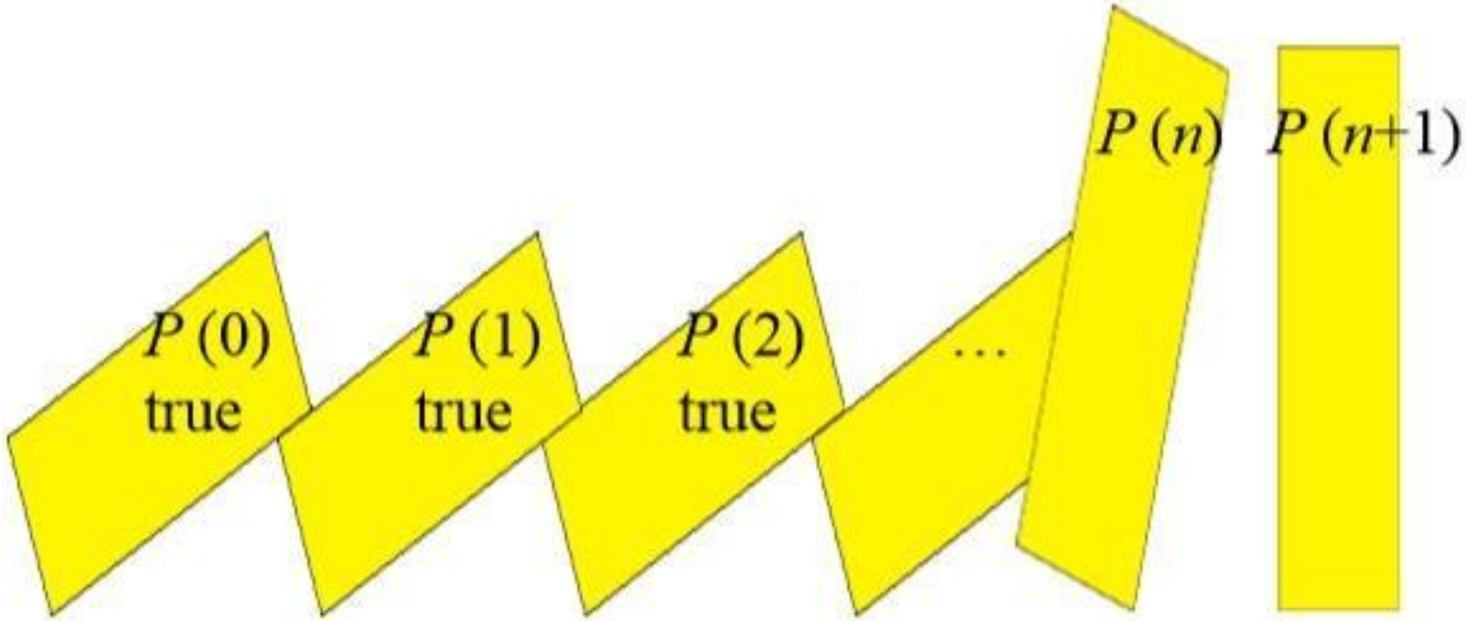

Mathematical Induction

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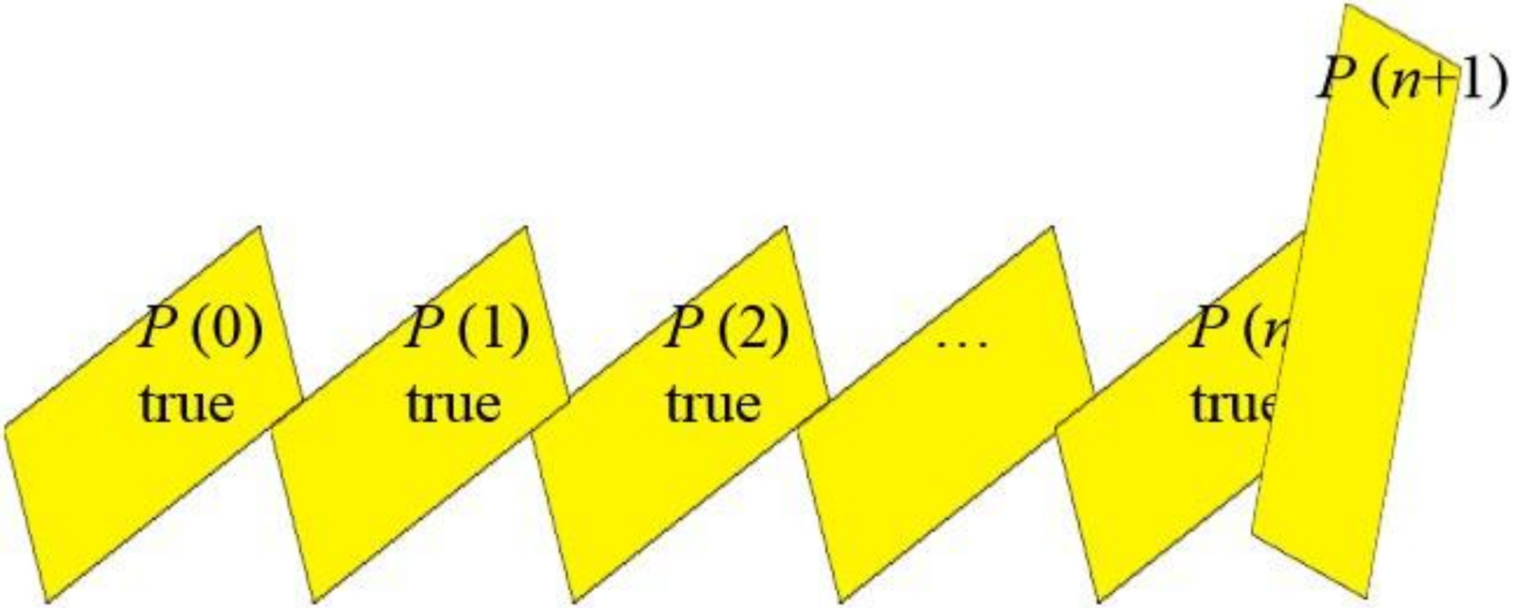

Mathematical Induction

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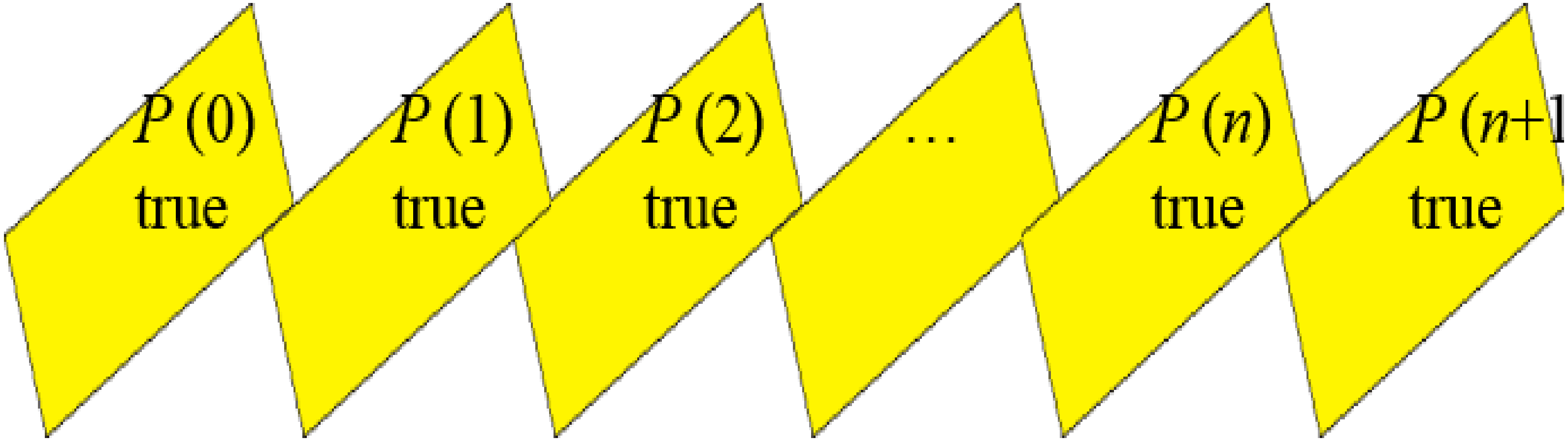


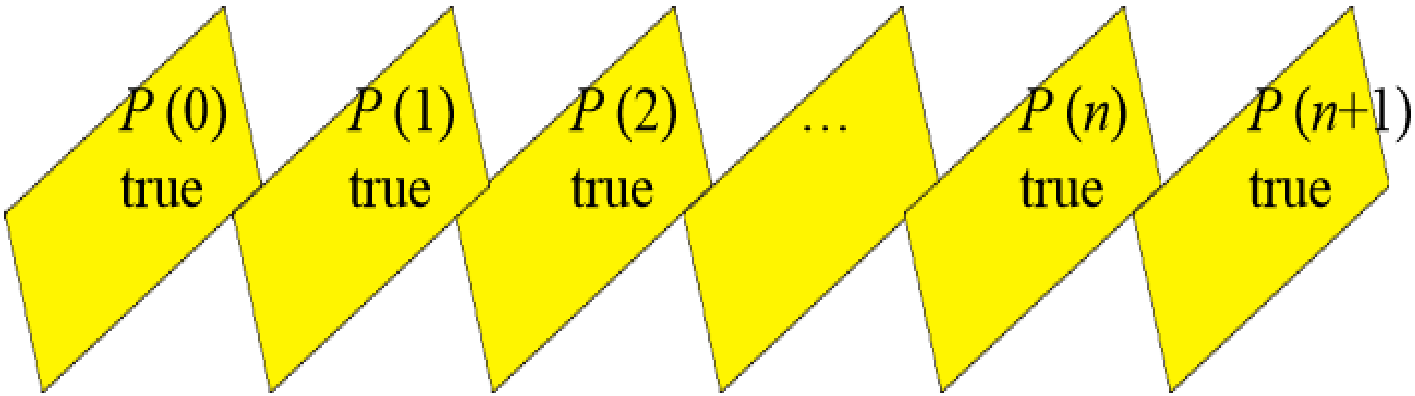
Mathematical Induction

Principle of Mathematical Induction:

If: 1) [***Basis Step***] $P(0)$ is true

2) [***Induction Step***]

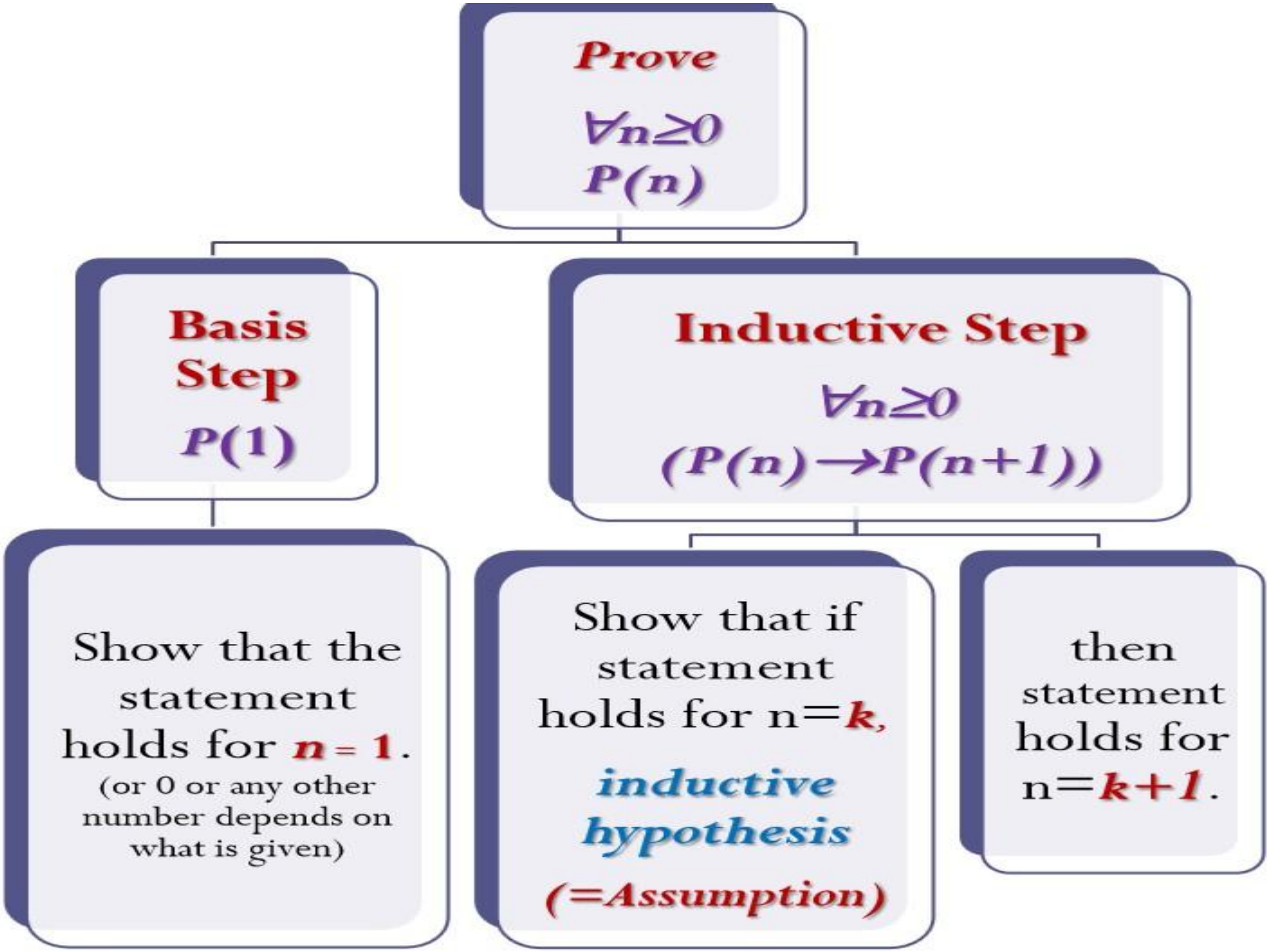
$\forall n \; P(n) \rightarrow P(n+1)$ is true



Then: $\forall n \; P(n)$ is true

This formalizes what occurred to dominos.

Mathematical Induction



Expressed as a rule of inference, this proof technique can be stated as

$$[P(1) \land \forall k\, P(k) \rightarrow P(k+1)\,] \rightarrow \forall n\, P(n)$$

Conjecturing and Proving Correct a Summation Formula

Example: Conjecture and prove correct a formula for the sum of the first *n* positive odd integers. Then prove your conjecture.

Solution: We have: $1 = 1, 1 + 3 = 4, 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16, 1 + 3 + 5 + 7 + 9 = 25.$

- *We can conjecture that the sum of the first n positive odd integers is n^2,*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

- *We prove the conjecture is proved correct with mathematical induction.*

1. ***BASIS STEP****: P(1) is true since $1^2 = 1$.*
2. ***INDUCTIVE STEP:*** *P(k) → P(k + 1) for every positive integer k.*

Assume the inductive hypothesis holds and then show that *P*(*k*) holds has well.

Inductive Hypothesis: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

- *So, assuming P(k), it follows that:*

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1) \quad (\textit{by the inductive hypothesis}) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

- *Hence, we have shown that P(k + 1) follows from P(k). Therefore the sum of the first n positive odd integers is n^2.*

Mathematical Induction Example-p.316

EG: Show that if n is a positive integer, then

$$1 + 2 + \dots\dots\dots + n = \frac{n(n+1)}{2}$$

Proof:

Show that if n is a positive integer, then:

$$1+2+\ldots+n=\frac{n(n+1)}{2}$$

Sol: Basis step: P(1) is true since $1 = 1(1+1)/2$

Inductive step: Assume P(k) holds so that:

$1+2+\ldots+k=k(k+1)/2$

show that P(k+1) is true ,i.e,

$1+2+\ldots+k+(k+1)=(k+1)[(k+1)+1]/2$

$=(k+1)(k+2)/2$ is true

Add (k+1) to both sides of the equation of P(k):

$1+2+\ldots+k+(k+1)=k(k+1)/2+(k+1)$

$=$

$k(k+1)/2+2(k+1)/2=(k+1)(k+2)/2$,

so P(k+1) is true

Mathematical Induction Example-p.318

EG: Use mathematical induction to show that

$$1+2+2^2 \ldots\ldots\ldots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Mathematical Induction Example-p.318

EG: Use mathematical induction to prove this formula for the sum of a finite number of terms of a **geometric progression:**

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is nonnegative integer.

5.3. Recursive Definitions

Recursive Definitions & Induction

Recursive definition and **inductive proofs** are **complement** to each others: a recursive definition usually gives rise to natural proofs involving the recursively defined sequence.

This follows from the format of a **recursive definition** as consisting of two parts:

1. **Initialization** –analogous to induction **base cases**
2. **Recursion** –analogous to **induction step**

In both induction and recursion, the domino analogy is useful.

Recursively Defined Functions

Definition: A *recursive* or *inductive definition* of a function consists of two steps.

1. ***BASIS STEP****: Specify the value of the function at zero.*
2. ***RECURSIVE STEP****: Give a rule for finding its value at an integer from its values at smaller integers.*

Recursively Defined Functions

Basis Step: **Specify** the **value** of the **function** at **zero**

Example 1: Suppose f is defined by:

Basis step: f(0) = 3

Recursive definition: **f (n+1) = 2 f(n)+3**

Find f(1), f(2), f(3) and f(4)?

f (n+1) = 2 f(n)+3

n=0, f(1) = 2f(0)+3 = 2*3+3=9

n=1, f(2) = 2f(1)+3 = 2*9+3=21

n=2, f(3) = 2f(2)+3 = 2*21+3=45

n=3, f(4) = 2f(3)+3 = 2*45+3=93

Recursive Step: Give a rule for **finding its value** at an **integer** from its values **at smaller integers.**

Example 2

The recursive definition:

$f(0) = 1$

$f(n) = n.f(n-1)$

Find $f(5)$?

$$f(n) = n.f(n-1)$$

n=1, $f(1)=1.f(0) = 1.1 = 1$

n=2, $f(2)=2.f(1)= 2.1= 2$

n=3, $f(3)=3.f(2)= 3.2= 6$

n=4, $f(4)=4.f(3)= 4.6= 24$

n=5, $f(5)=5.f(4)= 5 . 24 = 120$

OR in 1 step like following:

$$f(5)=5.f(4)=5.4.f(3)=5.4.3.f(2)=$$
$$5.4.3.2.f(1)=5.4.3.2.1.f(0) = 5.4.3.2.1.1 = 120$$

Example 3

The fibonacci numbers, $f_0, f_1, f_2, \ldots$ are

defined by the equations :

$f_0=0 \quad , f_1=1$

$f_n= f_{n-1}+f_{n-2} \quad \text{for } n=2, 3, 4, \ldots$

Find f_2, f_3 and f_4?

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$f_0=0 \quad , f_1=1$

$f_n= f_{n-1}+f_{n-2} \quad$ for n=2 , 3 ,4, ...

Find f_2, f_3 and f_4?

$f_n= f_{n-1}+f_{n-2}$

n=2, $f_2 = f_1+f_0 = 1+0=1$

n=3, $f_3 = f_2+f_1 = 1+1=2$

n=4, $f_4 = f_3+f_2 = 2+1 =3$

Example 4

Given: $f(0)=3$, $f(n+1)=2f(n)$, $n \geq 0$

Find $f(1)$, $f(2)$,………… $f(5)$

Example 4

<u>**Given**</u>: $f(0)=3$, $f(n+1)=2f(n)$, $n \geq 0$

Find f(1), f(2),........... f(5)

$$f(n+1)=2f(n)$$

$n=0$, $f(1) = 2f(0) = 2*3 = 6$

$n=1$, $f(2) = 2f(1) = 2*6 = 12$

$n=2$, $f(3) = 2f(2) = 2*12 = 24$

$n=3$, $f(4) = 2f(3) = 2*24 = 48$

$n=4$, $f(5) = 2f(4) = 2*48 = 96$

Example 5

Given: $f(n+1)=f^2(n)-2f(n)-2$, $f(0)=3$, $n \geq 0$

Find $f(1)$, $f(2)$

Example 5

Given: $f(n+1)=f^2(n)-2f(n)-2$, $f(0)=3$, $n\geq 0$

Find f(1), f(2)

$$f(n+1)=f^2(n)-2f(n)-2$$

n=0, $f(1) = f^2(0) - 2f(0)-2$

$= 3^2-2(3)-2=9-6-2=1$

n=1, $f(2) = f^2(1) -2f(1) -2$

$= 1^2-2(1)-2=1-2-2=-3$