Mathematical Induction 5.1 & 5.3

Chapter Summary

1. Mathematical Induction
2. Examples of Proof by Mathematical Induction
3. Recursive Definitions

Climbing an Infinite Ladder

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder. 2. If we can reach a particular rung of the ladder, then we can reach the next rung.

The principle of Mathematic induction: 1. Show we can reach the first step

2. Show that if we can reach step k, then we can reach step k+1

if we establish these 2 things, we know we can reach all the steps of the ladder.

This example motivates proof by mathematical induction.

Principle of Mathematical Induction (MI)

Principle of Mathematical Induction: To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete these steps:

1. **Basis Step:** Show that P(1) is true.
2. **Inductive Step:** Show that P(k) → P(k + 1) is true for all positive integers k.

Mathematical induction can be expressed as the rule of inference

$$(P(1) \land \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n),$$

where the domain is the set of positive integers.

Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point *b* where *b* is an integer. We will see examples of this soon.

Remembering How Mathematical

Induction Works - Domino Effect

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

…pposition that the …cked over.

We know that the first domino is knocked down, i.e., $P(1)$ is true .

Hence, all dominos are knocked over. $P(n)$ is true for all positive integers n.

Mathematical Induction

Suppose we have a sequence of propositions which we would like to prove:

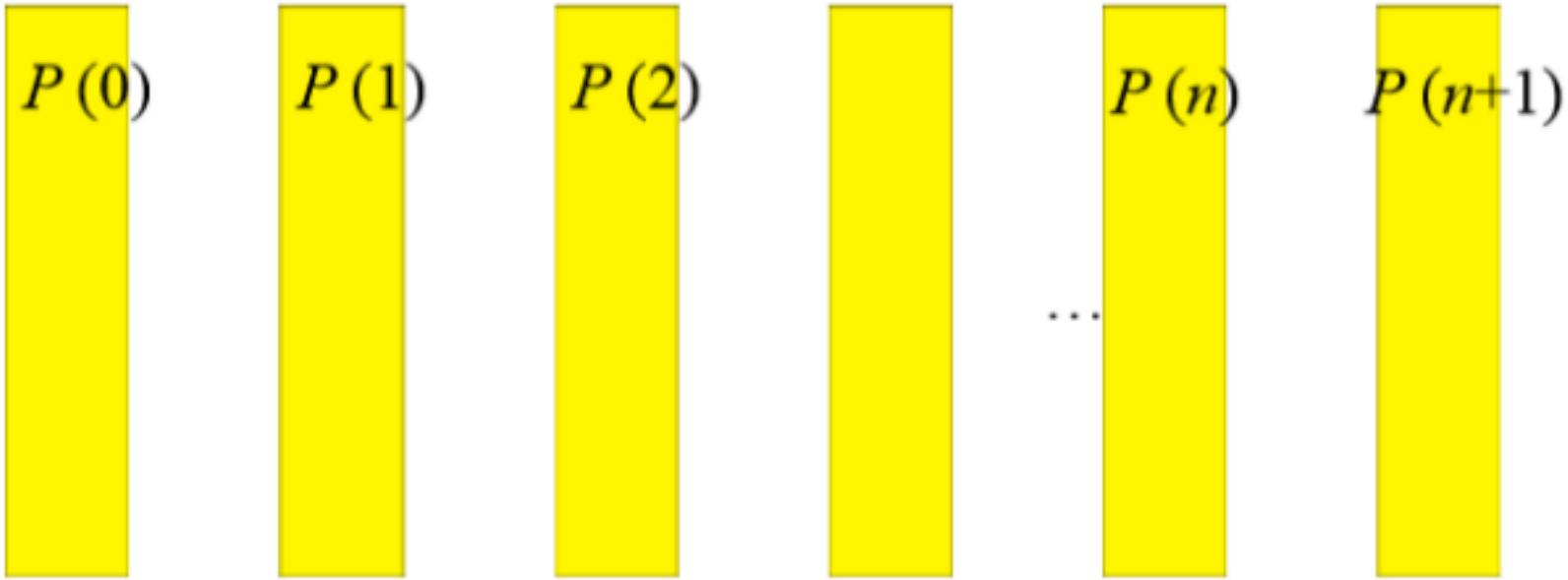
$P(0), P(1), P(2), P(3), P(4), \ldots P(n), \ldots$

EG: $P(n) =$

We can picture each proposition as a domino:

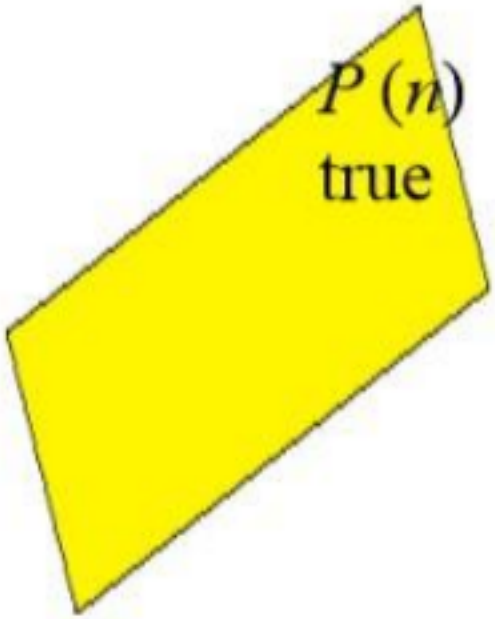
Mathematical Induction

So sequence of propositions is a sequence of dominos.



Mathematical Induction

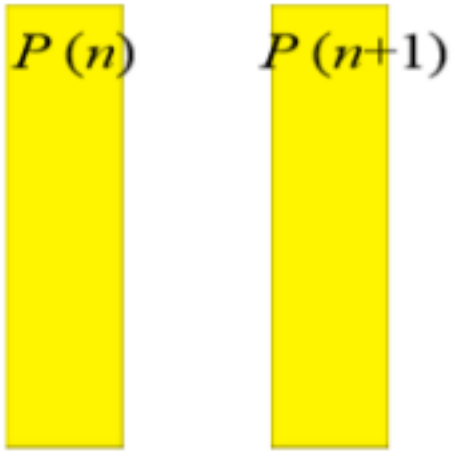
When the domino falls (to right), the corresponding proposition is considered true:



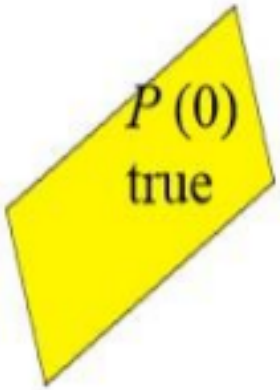
Mathematical Induction

Suppose that the dominos satisfy two constraints.

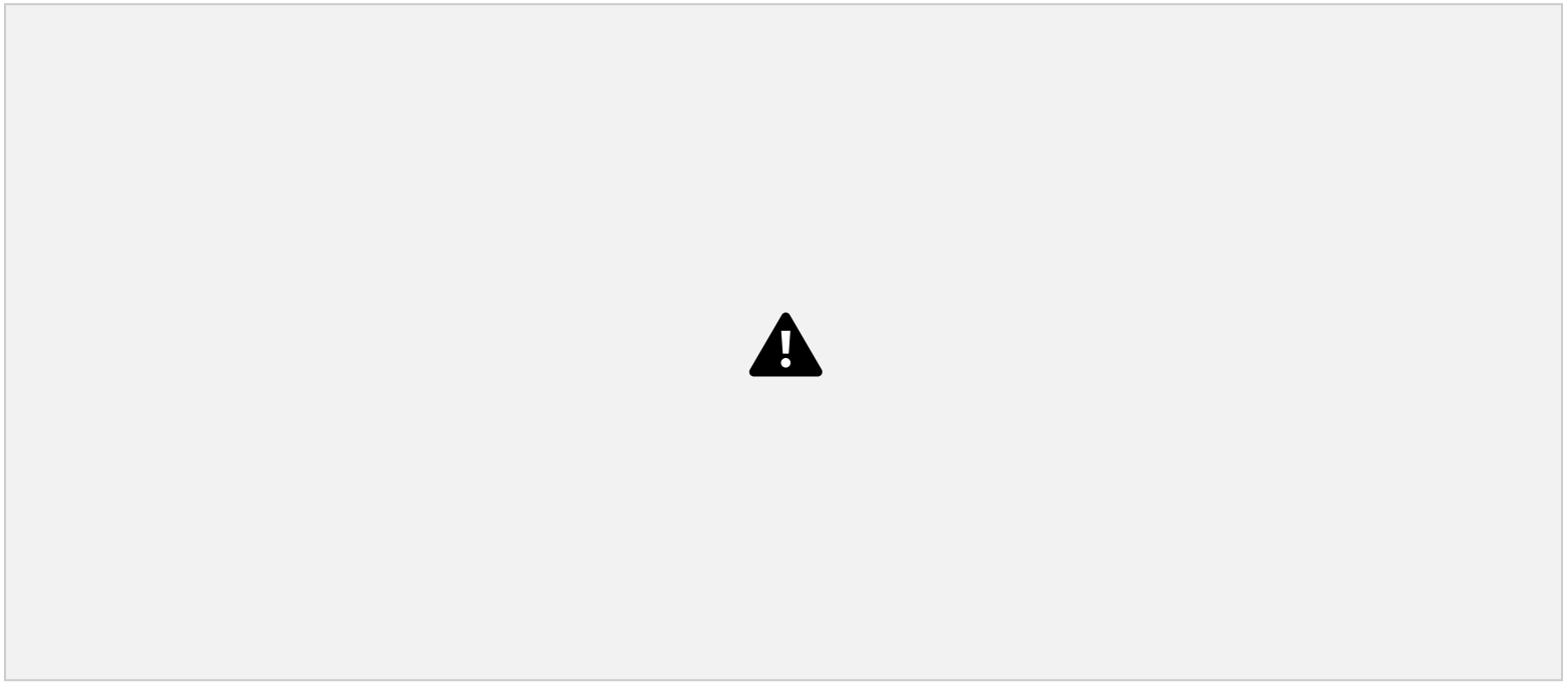
1) Well-positioned: If any domino falls (to right), next domino (to right) must fall also.



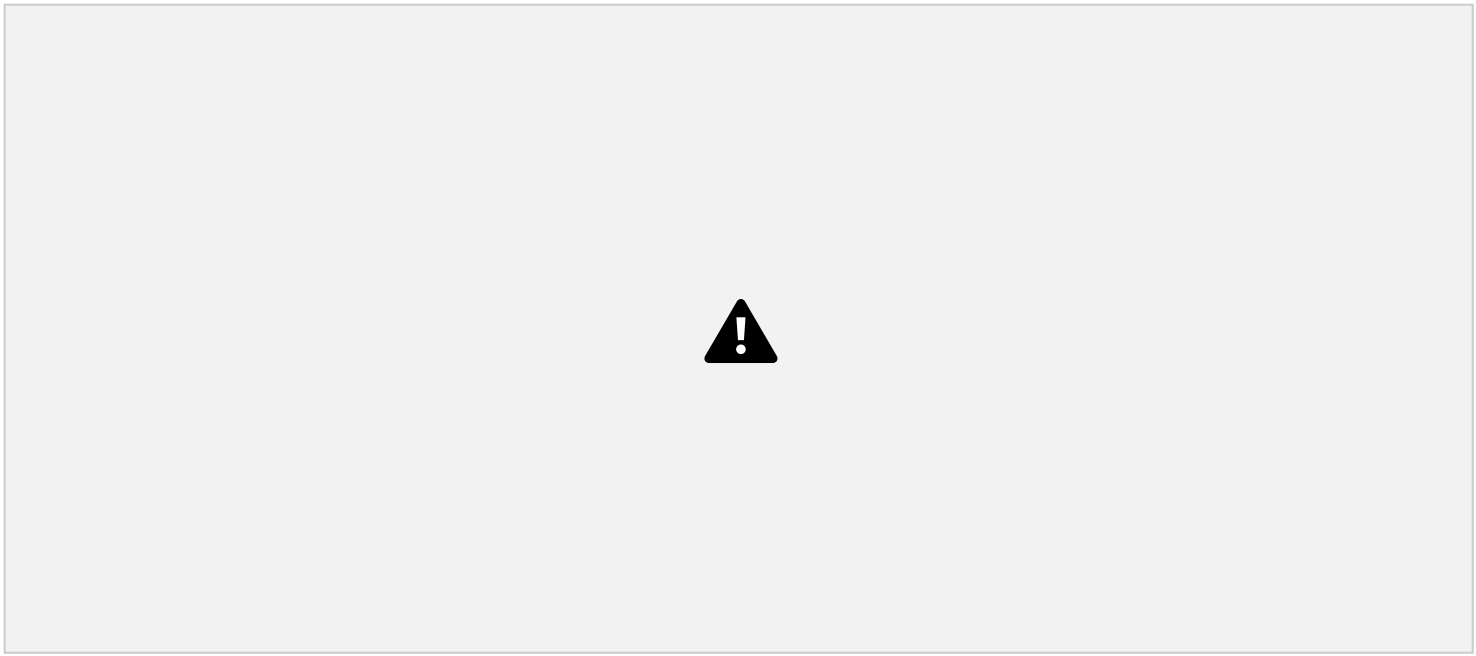
2) First domino has fallen to right



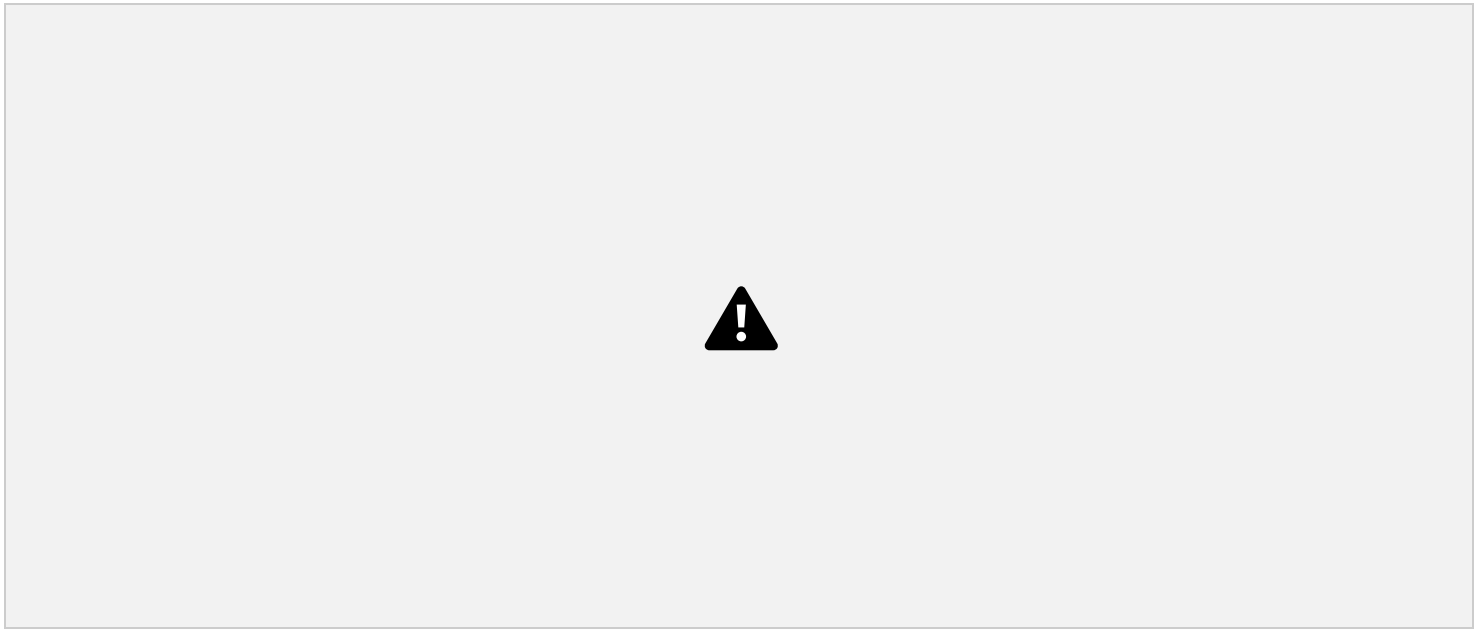
Mathematical Induction



Mathematical Induction



Mathematical Induction



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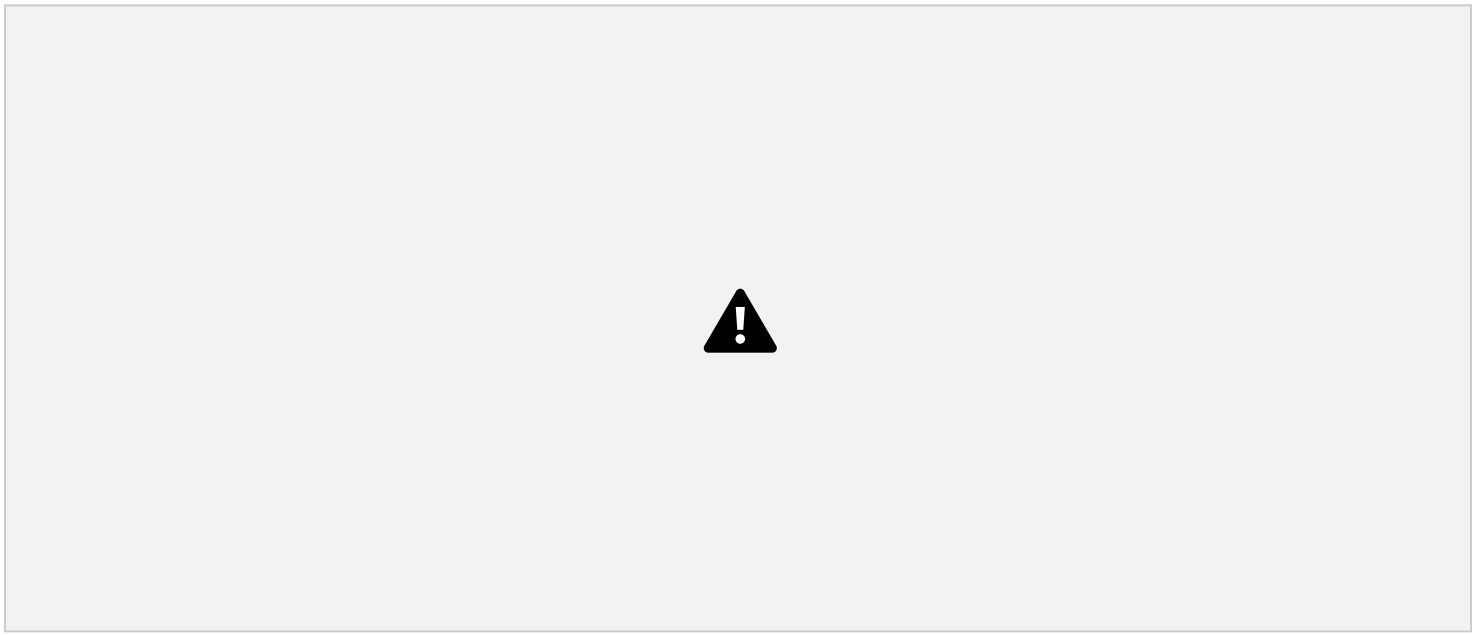

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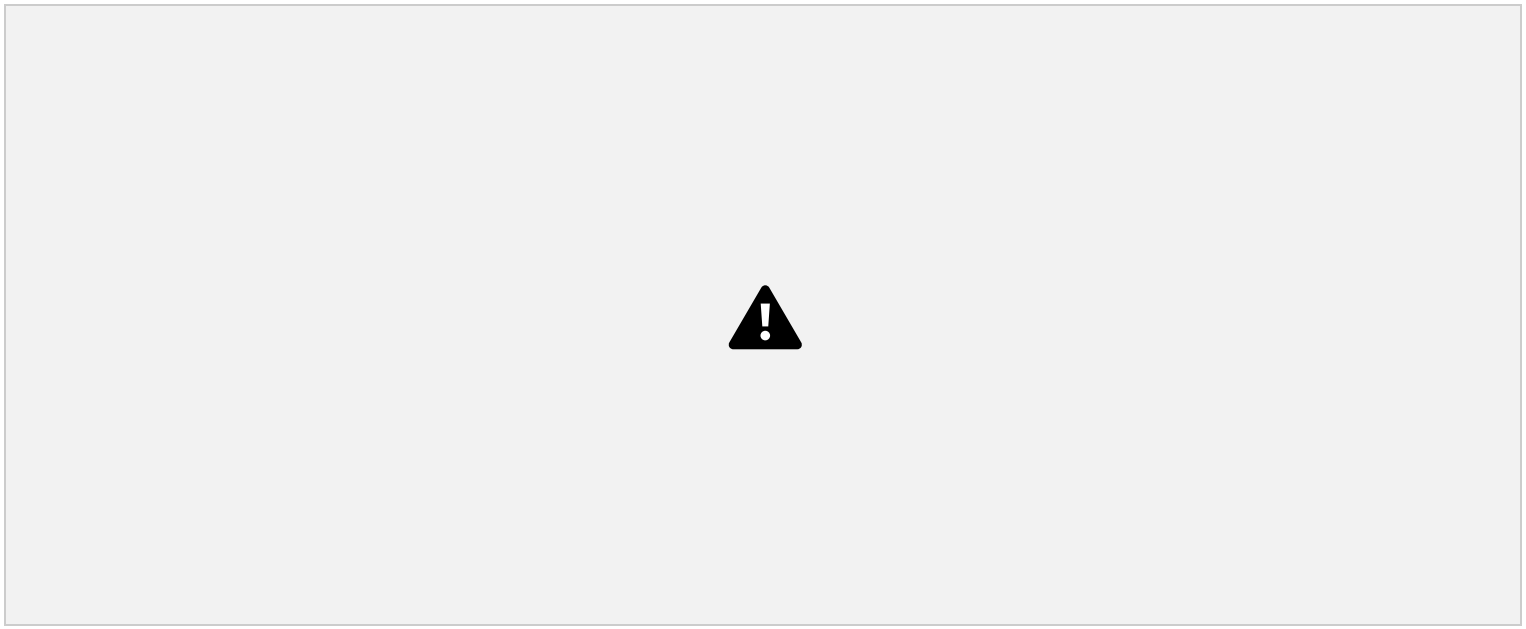

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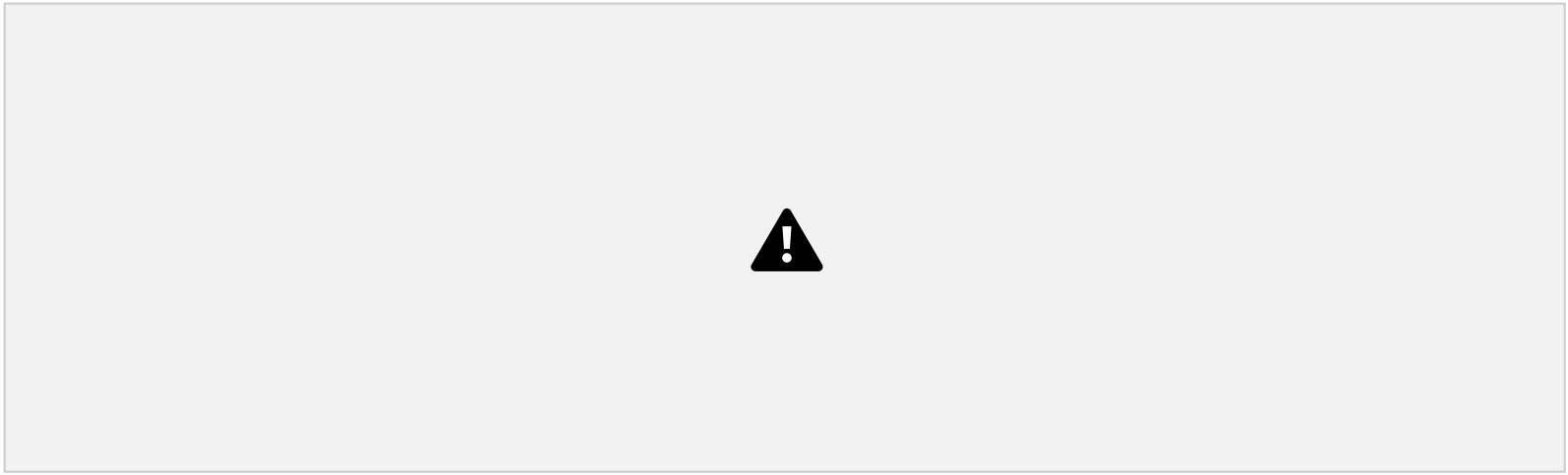

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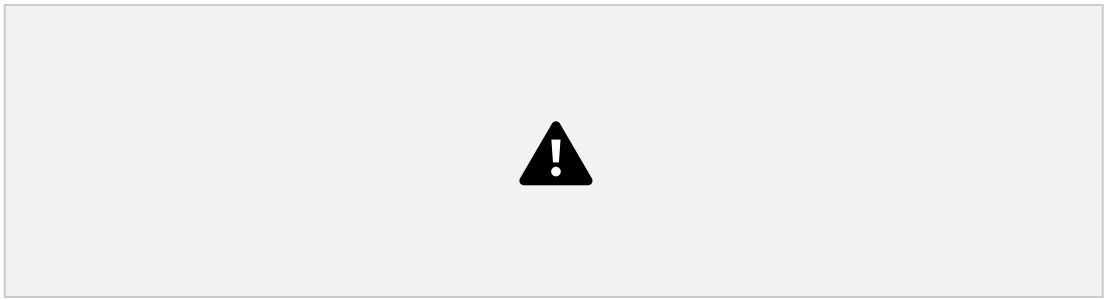
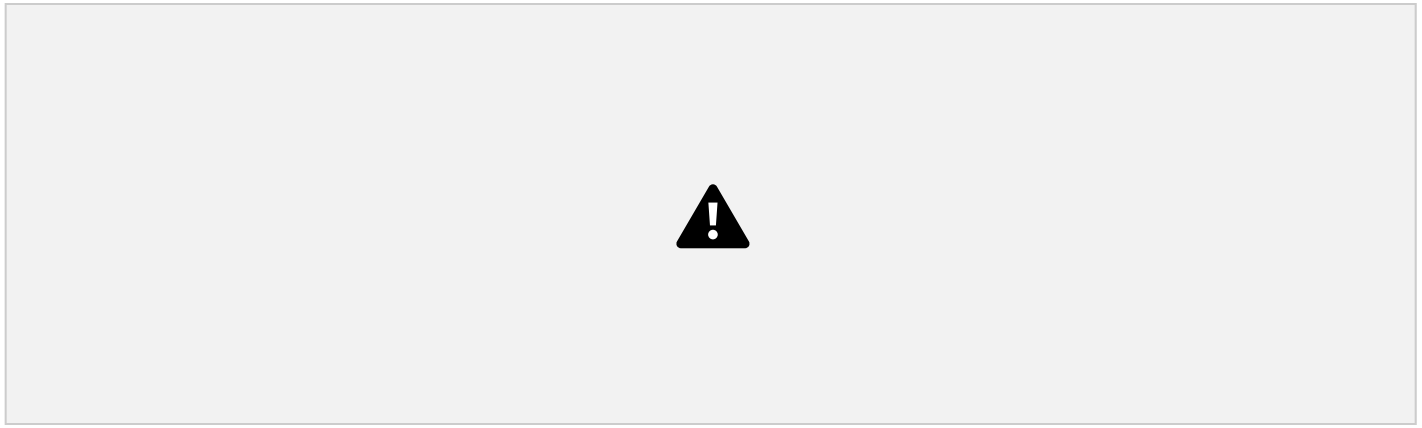
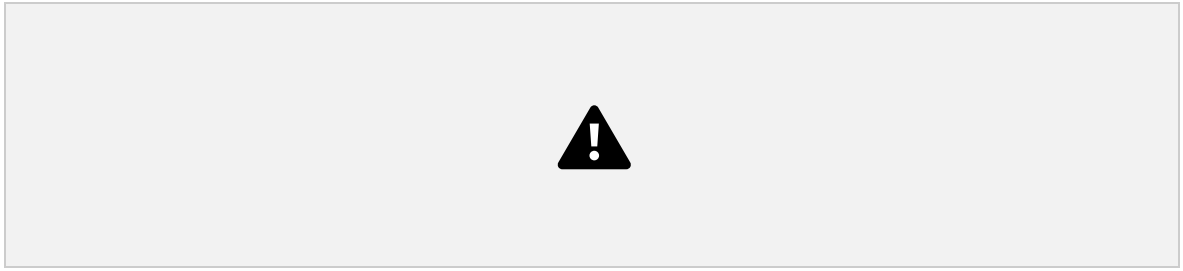




Mathematical Induction



Conjecturing and Proving Correct a Summation Formula

Example: Conjecture and prove correct a formula for the sum of the first *n* positive odd integers. Then prove your conjecture.

Solution: We have: $1 = 1, 1 + 3 = 4, 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16, 1 + 3 + 5 + 7 + 9 = 25.$ • *We can conjecture that the sum of the first n positive odd integers is* n^2, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

• *We prove the conjecture is proved correct with mathematical induction.*

1. ***BASIS STEP**: P(1) is true since* $1^2 = 1$.

2. INDUCTIVE STEP: *P(k) → P(k + 1) for every positive integer k.* Assume the inductive hypothesis holds and then show that *P*(*k*) holds has well. Inductive

Hypothesis: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

- *So, assuming P(k), it follows that:*

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1)$$
$$= k^2 + (2k + 1) \text{ (by the inductive hypothesis)}$$
$$= k^2 + 2k + 1$$
$$= (k + 1)^2$$

- *Hence, we have shown that P(k + 1) follows from P(k). Therefore the sum of the first n positive odd integers is n^2.*

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Proof:

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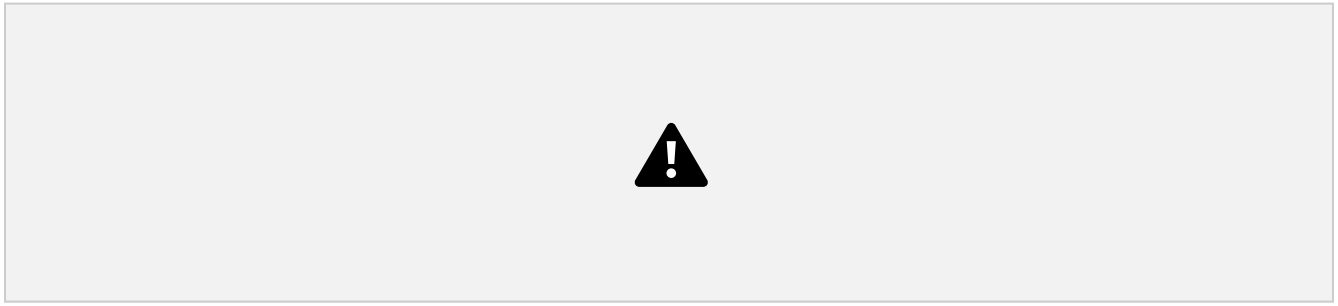
MathematicalInduction Example-p.318



5.3. Recursive Definitions

Recursive Definitions& Induction

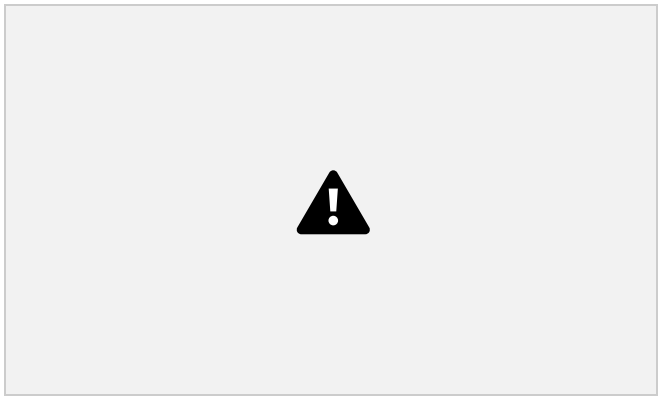
Recursively Defined Functions

Definition: A *recursive* or *inductive definition* of a function consists of two steps.

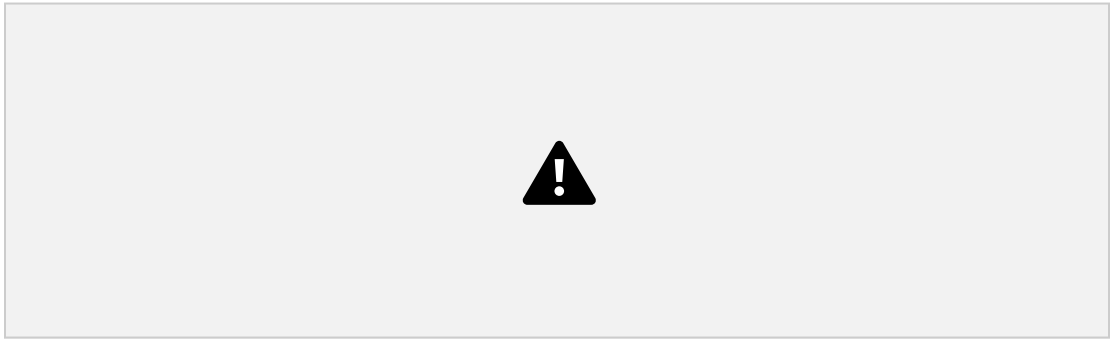
1. ***BASIS STEP****: Specify the value of the function at zero.*
2. ***RECURSIVE STEP****: Give a rule for finding its value at an integer from its values at*

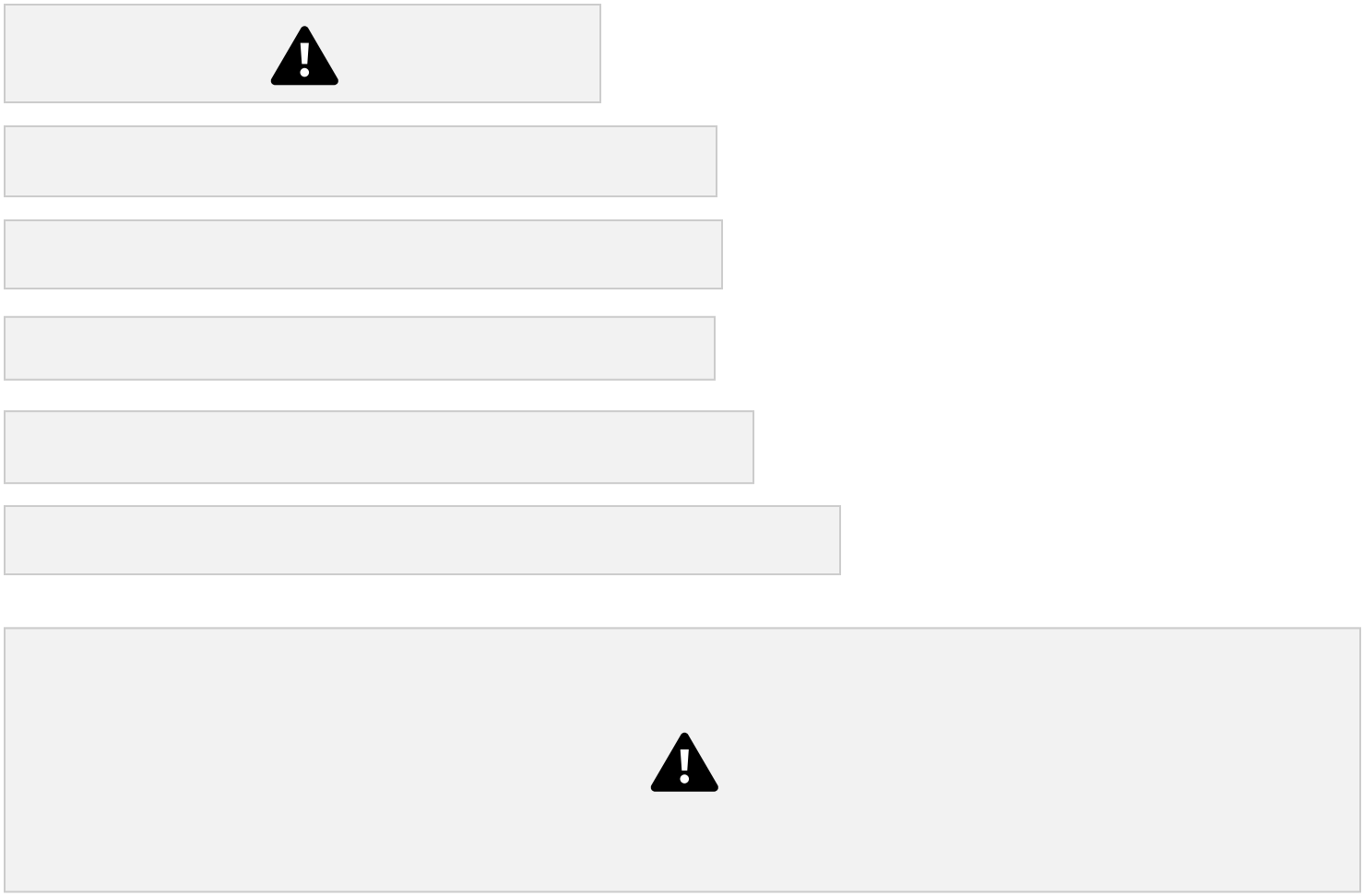
Recursively Defined Functions

Example1: Suppose f is defined by:



Example 2





Example 3

Example 3

Example 4

Example 4

Example 5

Example 5