Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

CHAPTER 2

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Set Cardinality
  - Countable Sets
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Matrices
  - Matrix Arithmetic

Sets

SECTION 2.1

Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called **naïve set theory**.

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The **objects** in a set are called the ***elements***, or ***members*** of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that **a is an element of the set A**.
- If a is not a member of A, write $a \notin A$

Describing Sets

- Roster Method
- Set-Builder Notation
- Interval Notation

Describing a Set: Roster Method

- List all the members of a set.
- $S = \{a,b,c,d\}$
- Order not important

$$S = \{a,b,c,d\} = \{b,c,a,d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

- Elipses (**...**) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a,b,c,d, \ldots\ldots,z\}$$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots\ldots,99\}$

- Set of all integers less than 0:

  $S = \{\ldots, -3,-2,-1\}$

Set-Builder Notation

- **Specify the property or properties** that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid \mathrm{P}(x)\}$$

- Example: $S = \{x \mid \mathrm{Prime}(x)\}$

- Positive rational numbers:

  $\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$

Interval Notation

$$[a,b] = \{x \mid a \leq x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

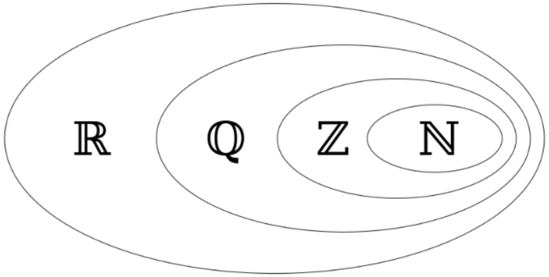
$$(a,b] = \{x \mid a < x \leq b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

closed *interval* **[**a,b**]**

open *interval* **(**a,b**)**

Some Important Sets

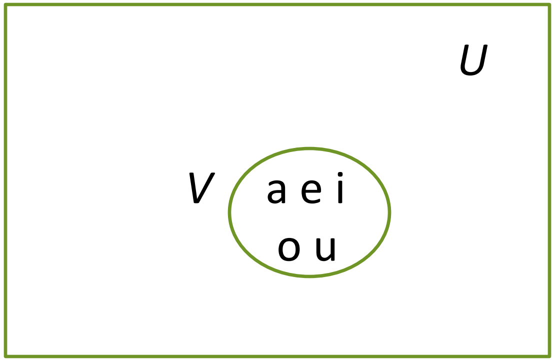


- N = natural numbers = {0,1,2,3….}
- Z = integers = {…,-3,-2,-1,0,1,2,3,…}
- Z^+ = positive integers = {1,2,3,…..}
- Q = set of rational numbers;
  - is any number that can be expressed as the **quotient or fraction** p/q of **two integers**
- R = set of real numbers;
  - This set includes **all rational numbers**, together with **all irrational numbers** (that is, numbers that cannot be rewritten as fractions, such as √2, π, e
- R+ = set of positive real numbers
- C = set of complex numbers.
  - C = {a + bi : a, b ∈ R}. For example, 1 + 2i ∈ C.

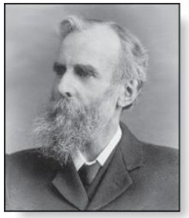
Universal Set and Empty Set

- The ***universal set U*** is the set **containing everything** currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The **empty set** is the set with **no elements**.
  - Symbolized ∅, but {} also used.

Venn Diagram

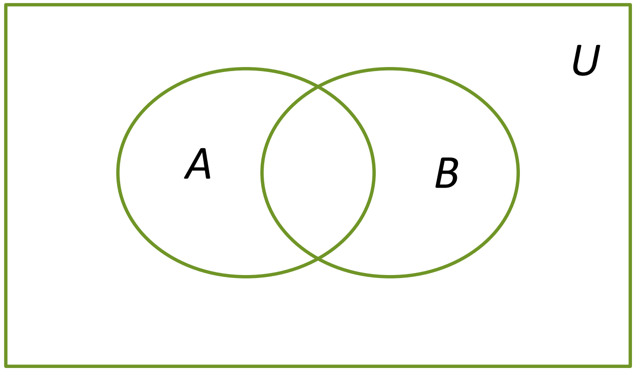


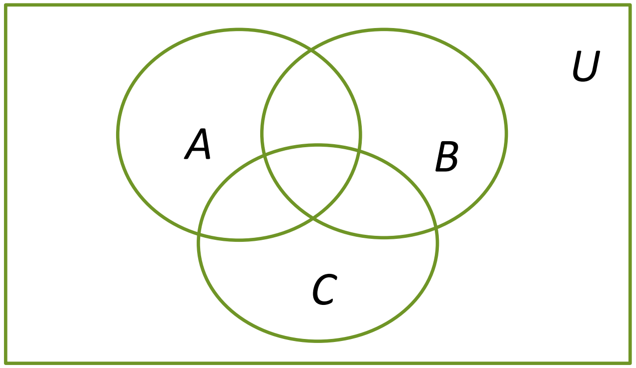
Venn Diagram



John Venn (1834-1923)
Cambridge, UK

- Named after the English mathematician John Venn, who introduced their use in 1881
- A useful geometric visualization tool (for 3 or less sets)
- The Universe U is the rectangular box
- Each set is represented by a circle and its interior
- Shade the appropriate region to represent the given set operation.





Some things to remember

- Sets can be elements of sets.

  $\{\,\{1,2,3\}\,,\,a\,,\,\{b,c\}\,\}$

  $\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\,\emptyset\,\}$

  $\emptyset = \{\}$

Set Equality

Definition: **Two** sets are ***equal*** if and only if they have the **same elements.**

- Therefore if A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set *A* is a *subset* of *B*, if and only if **every element of *A*** is **also an element of *B***.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set *S*.
2. Because $a \in S \to a \in S$, $S \subseteq S$, for every set *S*.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counter example to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets *A* and *B* are *equal*, denoted by *A* = *B*, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that *A* = *B* iff

$$\forall x[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$A \subseteq B$ and $B \subseteq A$

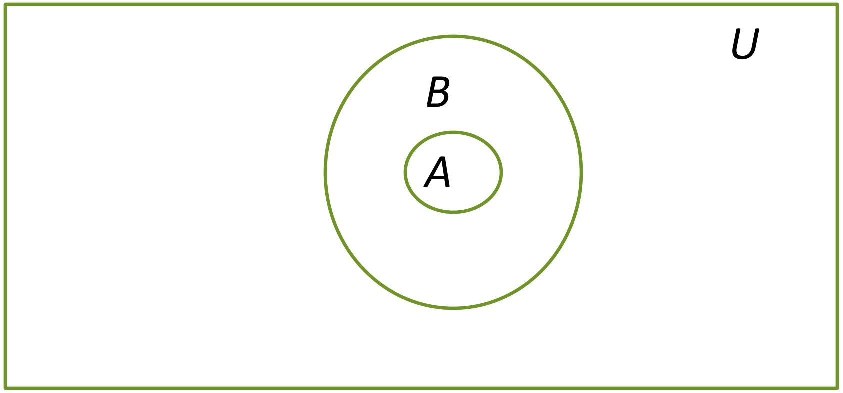
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

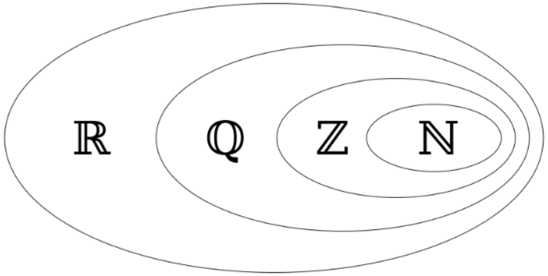
$$\forall x(x \in A \to x \in B) \land \exists x(x \in B \land x \notin A)$$

is true.

Venn Diagram



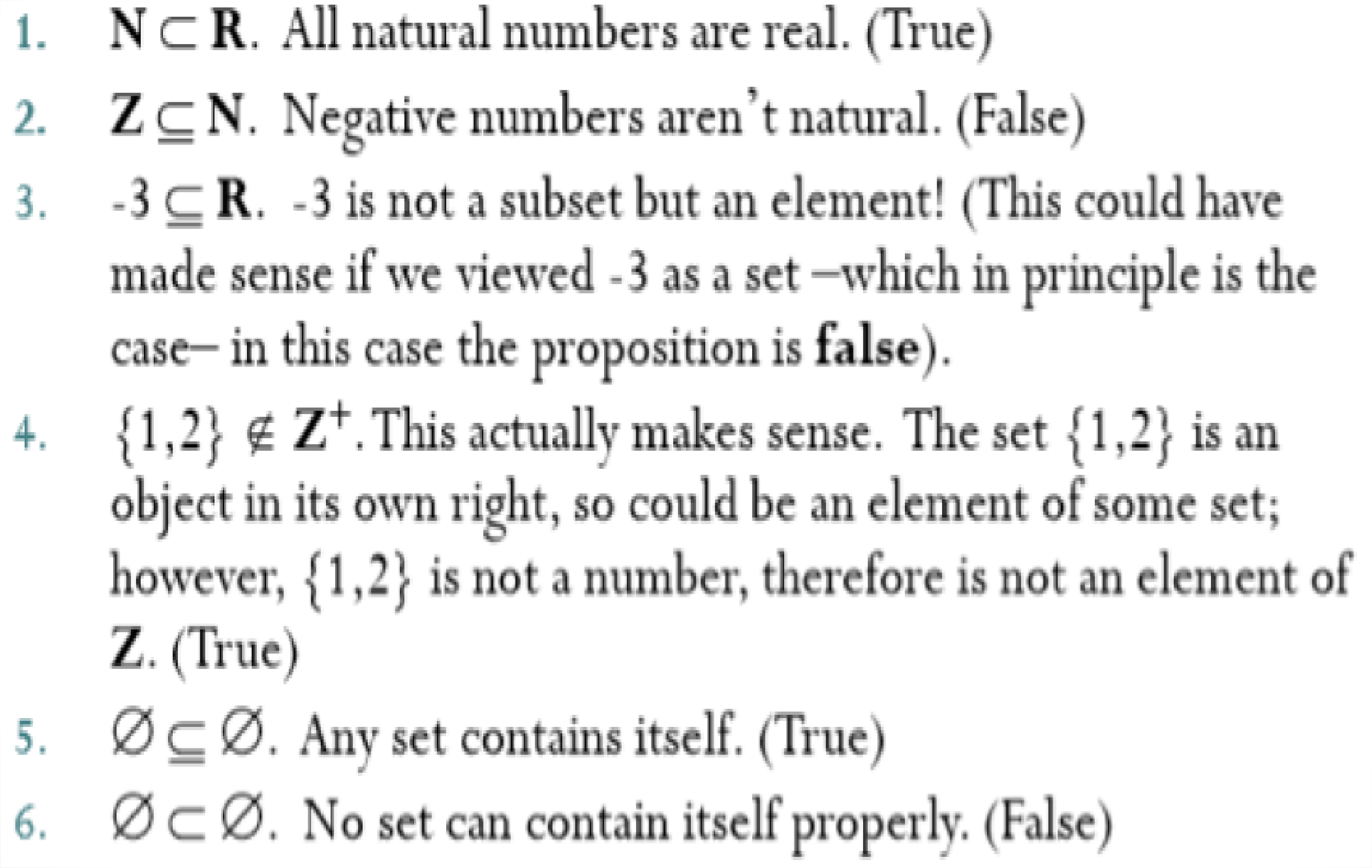
Subset Examples



Q: Which of the following are true:

1. $\mathbf{N} \subset \mathbf{R}$ (True)
2. $\mathbf{Z} \subseteq \mathbf{N}$ (False)
3. $-3 \subseteq \mathbf{R}$ (False)
4. $\{1,2\} \not\subseteq \mathbf{Z}^{+}$ (True)
5. $\emptyset \subseteq \emptyset$ (True)
6. $\emptyset \subset \emptyset$ (False)

Subset Examples



1. $\mathbf{N} \subset \mathbf{R}$. All natural numbers are real. (True)
2. $\mathbf{Z} \subseteq \mathbf{N}$. Negative numbers aren't natural. (False)
3. $-3 \subseteq \mathbf{R}$. -3 is not a subset but an element! (This could have made sense if we viewed -3 as a set –which in principle is the case– in this case the proposition is **false**).
4. $\{1,2\} \notin \mathbf{Z}^+$. This actually makes sense. The set $\{1,2\}$ is an object in its own right, so could be an element of some set; however, $\{1,2\}$ is not a number, therefore is not an element of $\mathbf{Z}$. (True)
5. $\varnothing \subseteq \varnothing$. Any set contains itself. (True)
6. $\varnothing \subset \varnothing$. No set can contain itself properly. (False)

Set Cardinality

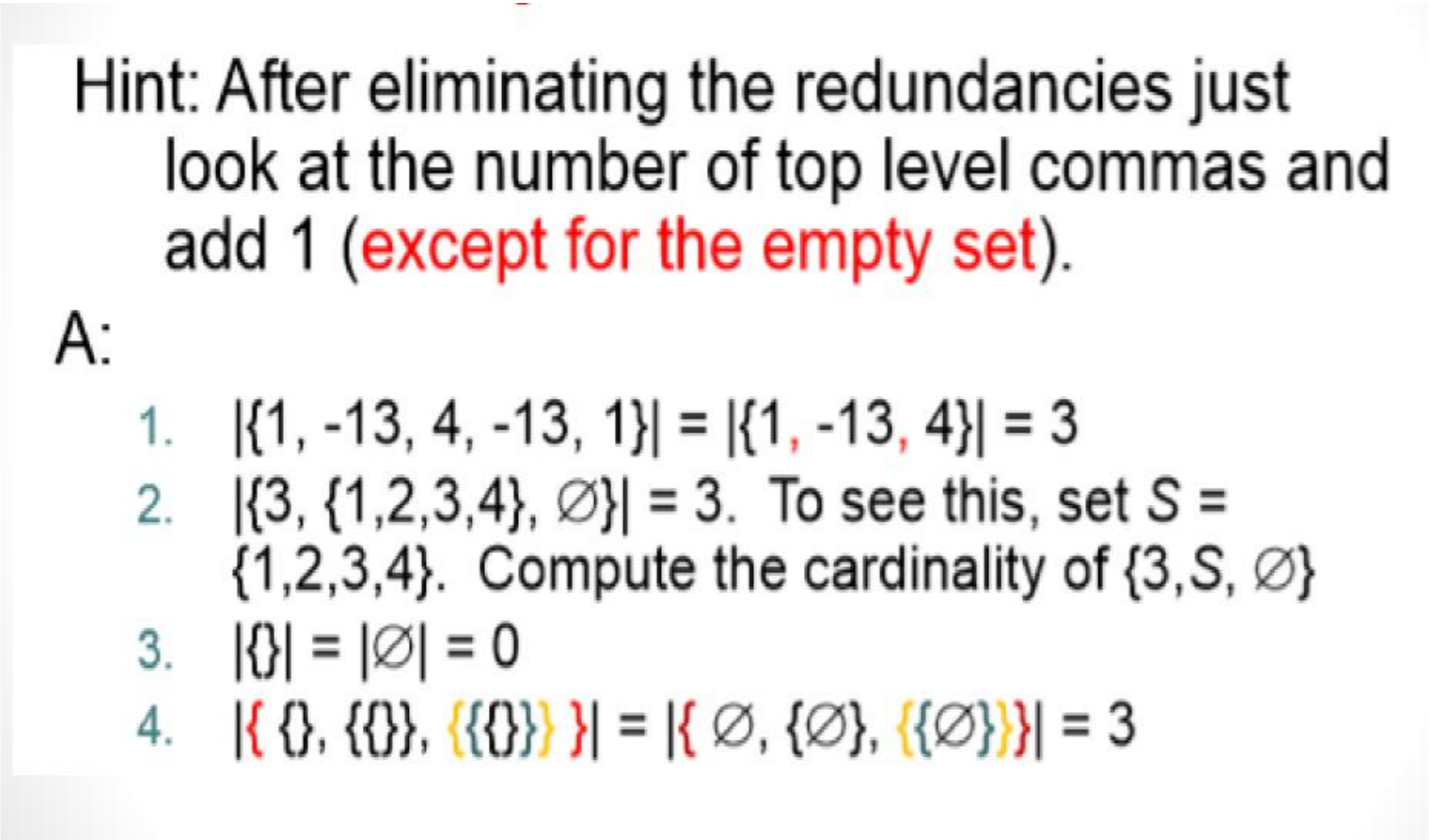
Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is ***finite set*** and that n is the **cardinality** of S. Otherwise it is *infinite*.

Definition: The *cardinality* of a **finite set** *A,* denoted by **|*A*|**, is the number of (distinct) elements of *A*.

Examples:

1. $|\varnothing| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\varnothing\}| = 1$
5. The set of integers is infinite.

Set Cardinality



Hint: After eliminating the redundancies just look at the number of top level commas and add 1 (except for the empty set).

A:

1. $|\{1, -13, 4, -13, 1\}| = |\{1, -13, 4\}| = 3$
2. $|\{3, \{1,2,3,4\}, \varnothing\}| = 3$. To see this, set $S = \{1,2,3,4\}$. Compute the cardinality of $\{3, S, \varnothing\}$
3. $|\{\}| = |\varnothing| = 0$
4. $|\{\ \{\}, \{\{\}\}, \{\{\{\}\}\}\ \}| = |\{\ \varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}| = 3$

Power Sets

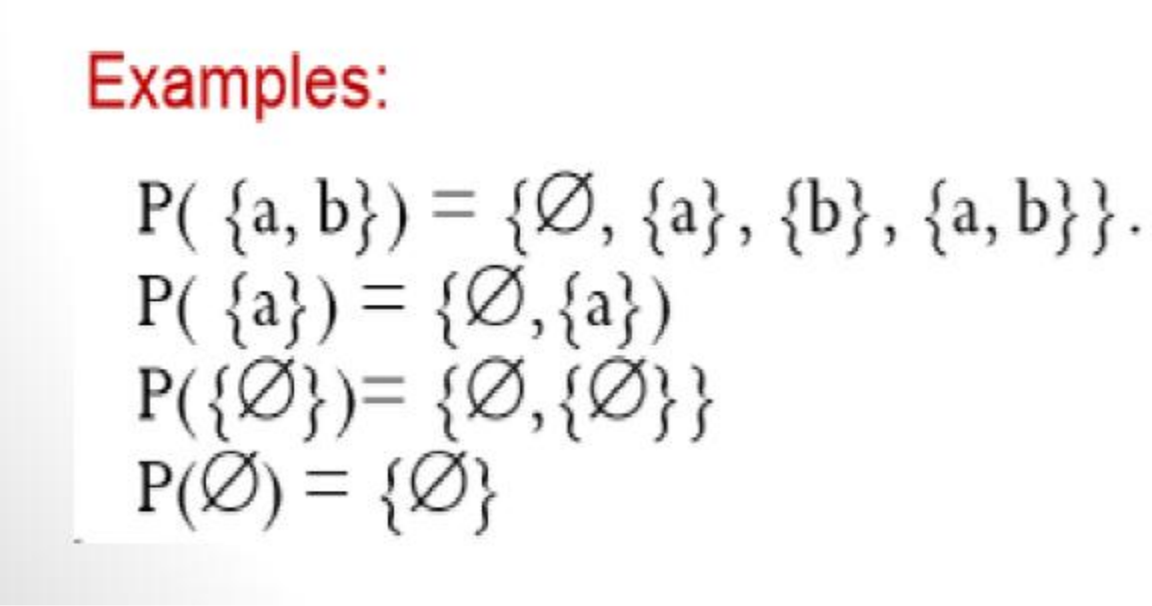
Definition: The set of all subsets of a set *A*, denoted P(*A*), is called the *power set* of *A*.

Example: If *A* = {a,b} then

P(A) = {ø, {a},{b},{a,b}}

- If a set has ***n* elements**, then the cardinality of the power set is $\mathbf{2^{n}}$. (In Chapters 5 and 6, we will discuss different ways to show this.)

Power Sets



Examples:

$$P(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

$$P(\{a\}) = \{\varnothing, \{a\})$$

$$P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$$

$$P(\varnothing) = \{\varnothing\}$$

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- Repetition and order do matter with n-tuples.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.
- Example:
  - $(1,2) \neq (2,1) \neq (1,2,1)$

Cartesian Product

René Descartes
(1596-1650)

Definition:

The *Cartesian Product* of two sets *A* and *B*, denoted by *A* × *B* is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **Definition**: A subset *R* of the Cartesian product *A* × *B* is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

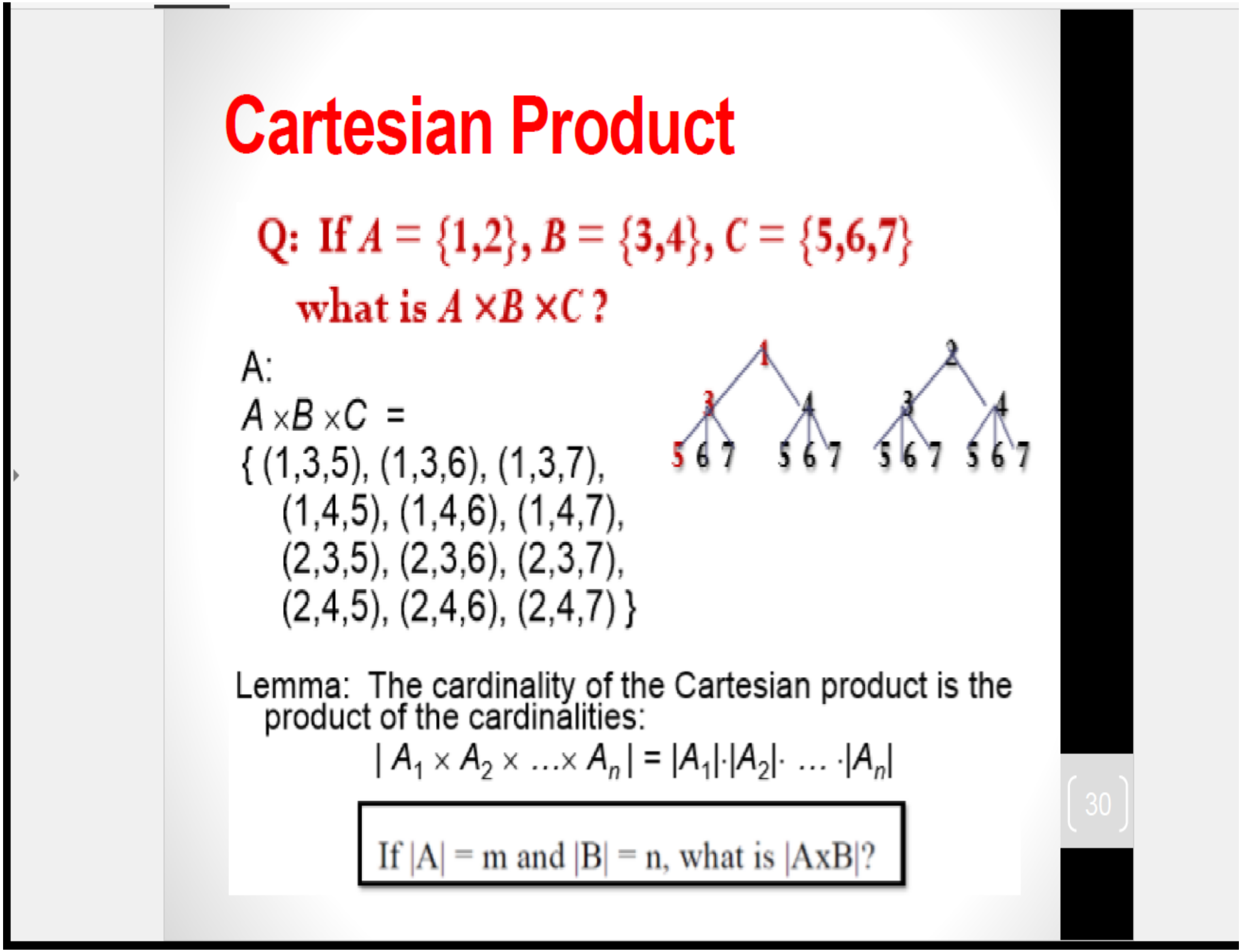
Cartesian Product

Definition: The cartesian products of the sets A_1,A_2,……,A_n, denoted by $A_1 \times A_2 \times …… \times A_n$, is the set of ordered *n*-tuples (a_1,a_2,……,a_n) where a_i belongs to A_i for *i* = 1, … n.

$$A_1 \times A_2 \times \cdots \times A_n = \\ \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is *A* × *B* × C where *A* = {0,1}, *B* = {1,2} and *C* = {0,1,2}

Solution: *A* × *B* × C **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}



Cartesian Product

Q: If $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6,7\}$ what is $A \times B \times C$?

A:

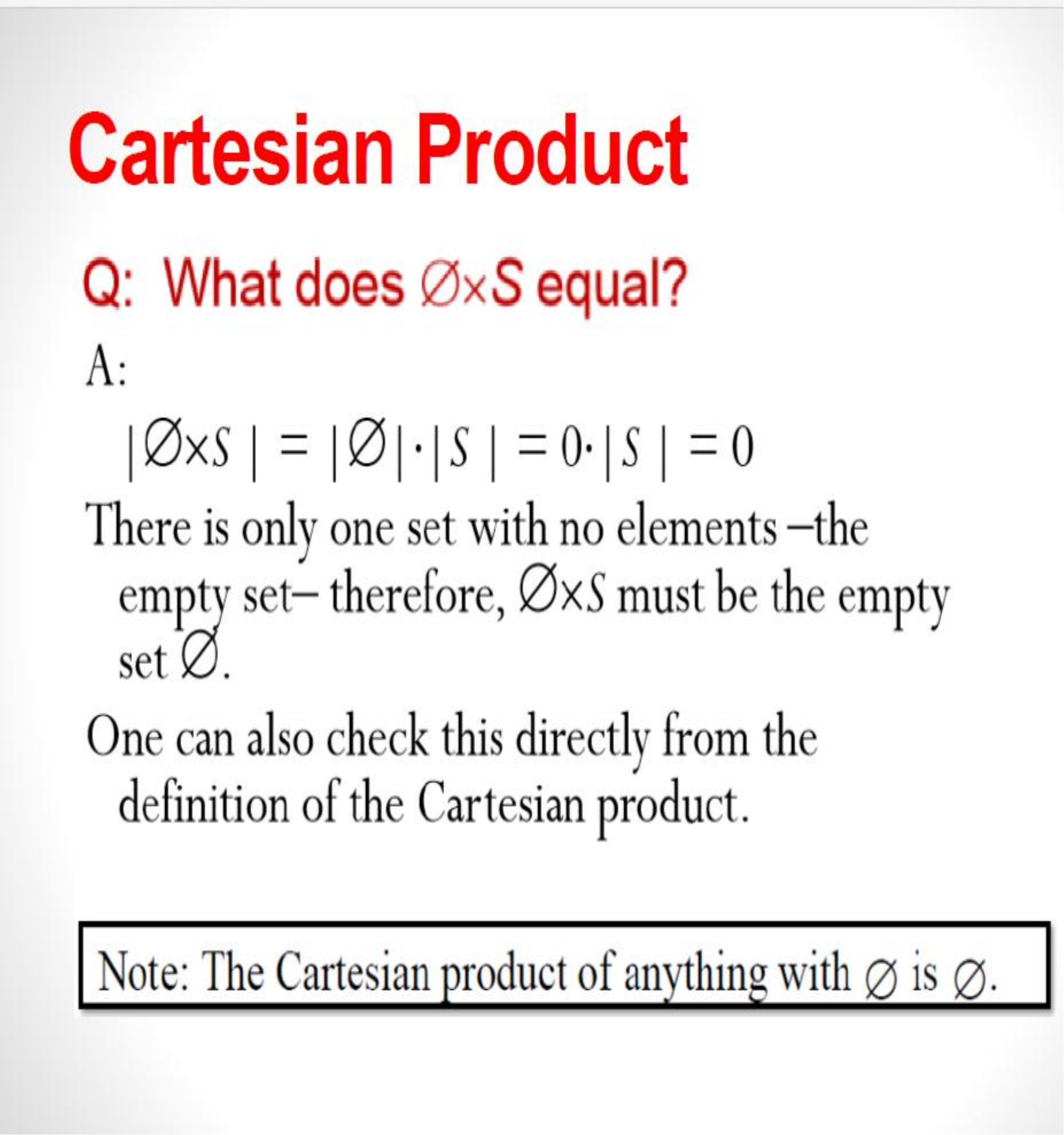
$A \times B \times C$ =

{ (1,3,5), (1,3,6), (1,3,7),
(1,4,5), (1,4,6), (1,4,7),
(2,3,5), (2,3,6), (2,3,7),
(2,4,5), (2,4,6), (2,4,7) }

Lemma: The cardinality of the Cartesian product is the product of the cardinalities:

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1|\cdot|A_2|\cdot \ldots \cdot|A_n|$$

If |A| = m and |B| = n, what is |AxB|?



Cartesian Product

Q: What does $\emptyset \times S$ equal?

A:

$$|\emptyset \times S| = |\emptyset| \cdot |S| = 0 \cdot |S| = 0$$

There is only one set with no elements –the empty set– therefore, $\emptyset \times S$ must be the empty set $\emptyset$.

One can also check this directly from the definition of the Cartesian product.

Note: The Cartesian product of anything with $\emptyset$ is $\emptyset$.

Cartesian Products of Sets

- For sets A, B, their *Cartesian product* $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.
- *E.g.* $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$
- Note that for finite A, B, $|A \times B| = |A|\,|B|$.
- Note that the Cartesian product is *not* commutative: $A \times B \neq B \times A$.
- *E.g.* $\{1,2\} \times \{a,b\} = \{(1,a),(1,b),(2,a),(2,b)\}$

Truth Sets of Quantifiers

- Given a predicate *P* and a domain *D*, we define the ***truth set*** of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Example**: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is “|*x*| = 1” is the set {-1,1}

Set Operations

SECTION 2.2

Section Summary

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Boolean Algebra

- Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*. This is discussed in Chapter 12.
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set *U*. All sets are assumed to be subsets of *U*.

Union

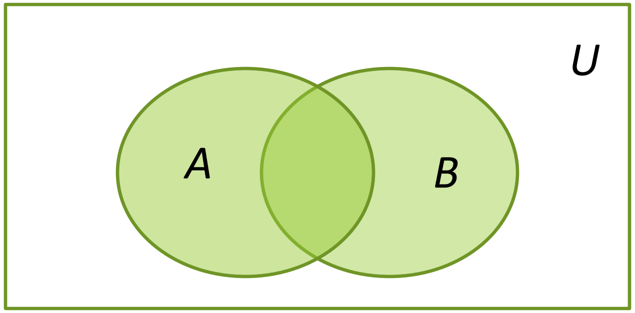
- **Definition**: Let *A* and *B* be sets. The ***union*** of the sets *A* and *B*, denoted by ***A* ∪ *B***, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The ***intersection*** of sets *A* and *B*, denoted by ***A* ∩ *B*,** is

$$\{x \mid x \in A \land x \in B\}$$

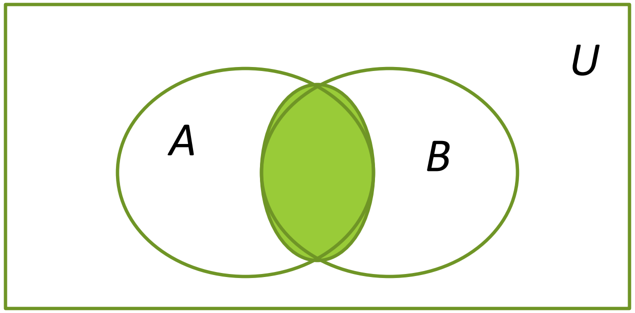
- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.
- **Example**: What is {1,2,3} ∩ {3,4,5} ?

Solution: {3}

- **Example:** What is {1,2,3} ∩ {4,5,6} ?

Solution: ∅

Venn Diagram for *A* ∩*B*



Complement

Definition: If A is a set, then the **complement** of the A (with respect to U), denoted by $\bar{A}$ is the set ***U* - *A***

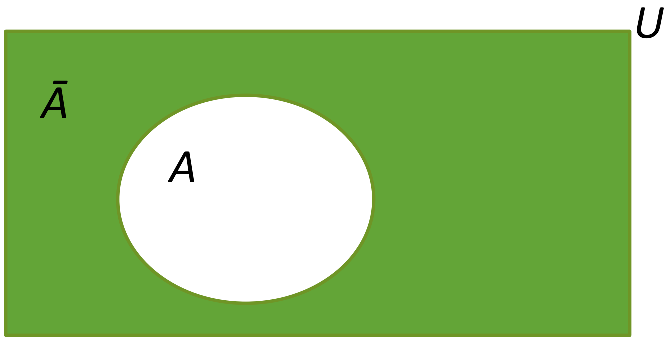
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

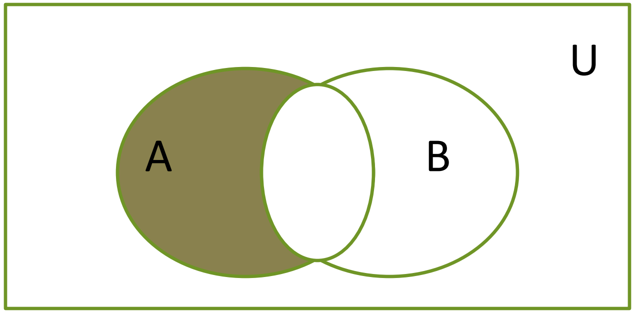
Venn Diagram for Complement



Difference

- **Definition**:
- Let A and B be sets. The ***difference*** of A and B, denoted by ***A* – *B***, is the set containing the **elements of *A* that are not in *B***.
- The difference of A and B is also called **the complement of *B* with respect to *A*.**

$$A - B = \{x \mid x \in \mathrm{A} \wedge x \notin B\} = A \cap \overline{B}$$

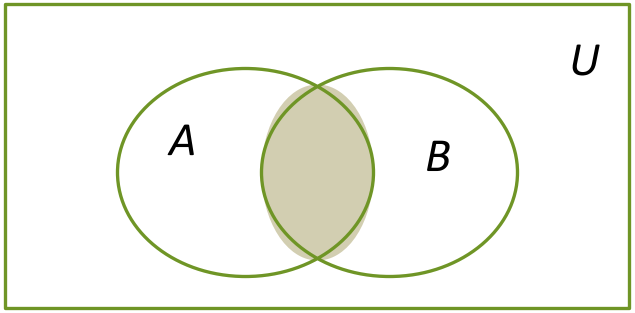


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

- **Example**: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by is the set $A \oplus B$ is the set $(A - B) \cup (B - A)$

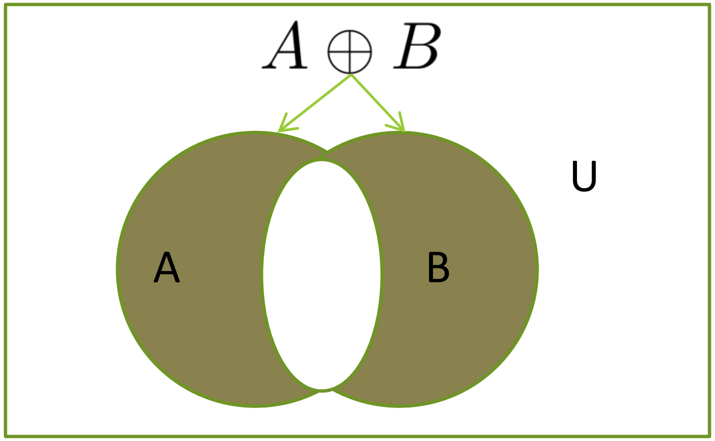
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is: $A \oplus B$

- **Solution**:
  1. $A - B = \{1,2,3\}$
  2. $B - A = \{6,7,8\}$
  3. $A \oplus B = \{1,2,3,6,7,8\}$



Venn Diagram

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways **to prove set identities**:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set-builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Continued on next slide →

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{aligned}
\overline{A \cap B} &= \{x | x \notin A \cap B\} && \text{by defn. of complement} \\
&= \{x | \neg(x \in (A \cap B))\} && \text{by defn. of does not belong symbol} \\
&= \{x | \neg(x \in A \wedge x \in B\} && \text{by defn. of intersection} \\
&= \{x | \neg(x \in A) \vee \neg(x \in B)\} && \text{by 1st De Morgan law} \\
& && \text{for Prop Logic} \\
&= \{x | x \notin A \vee x \notin B\} && \text{by defn. of not belong symbol} \\
&= \{x | x \in \overline{A} \vee x \in \overline{B}\} && \text{by defn. of complement} \\
&= \{x | x \in \overline{A} \cup \overline{B}\} && \text{by defn. of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of notation}
\end{aligned}
$$

Membership Table

Example: Construct a membership table to show that the **distributive law** holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	**B**	**C**	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- Let A_1, A_2 ,…, A_n be an indexed collection of sets.

  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

  These are well defined, since union and intersection are associative.

- For i = 1,2,…, let A_i = {i, i + 1, i + 2, ….}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

Functions

SECTION 2.3

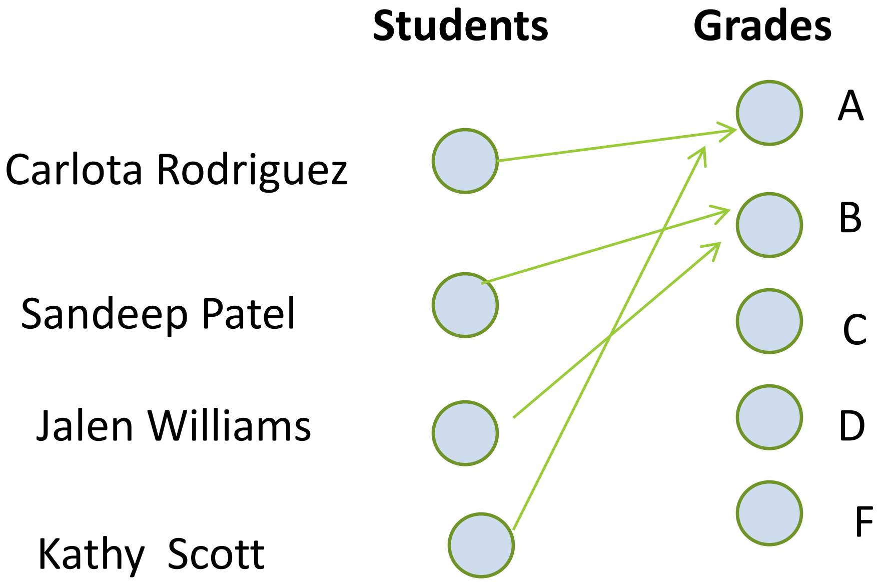
Section Summary

- Definition of a Function.
  - Domain, Cdomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (optional)

Functions

Definition: Let *A* and *B* be nonempty sets. A *function* f from *A* to *B*, denoted **f: *A* → *B*** is an assignment of each element of *A* to exactly one element of *B*. We write $\mathrm{f}(a) = b$ if *b* is the unique element of *B* assigned by the function f to the element *a* of *A*.

- **Functions** are sometimes called *mappings* or *transformations*.



Students

Grades

Carlota Rodriguez

Sandeep Patel

Jalen Williams

Kathy Scott

A

B

C

D

F

Functions

- A **function** f: *A* → *B* can also be defined as **a subset of *A*×*B* (a relation).** This subset is restricted to be a relation where no two elements of the relation have the same first element.

- Specifically, a function f from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element *a*∈ *A*.

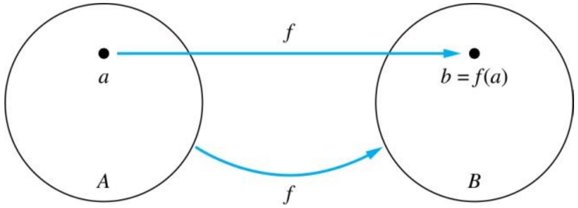
$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

and

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \rightarrow y_1 = y_2]$$

Functions

Given a function f: $A \to B$:



- We say ***f* maps *A* to *B*** *or*

  ***f* is a *mapping* from *A* to *B*.**

- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.
- If $f(a) = b$,
  - then *b* is called the *image* of *a* under *f*.
  - *a* is called the *preimage* of *b.*
- The range of *f* is the set of all images of points in **A** under *f*. We denote it by $f(A)$.
- **Two functions are *equal* when** they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Assignment of Grades in a Discrete Mathematics Class example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

f(a) = ? z

The image of d is ? z

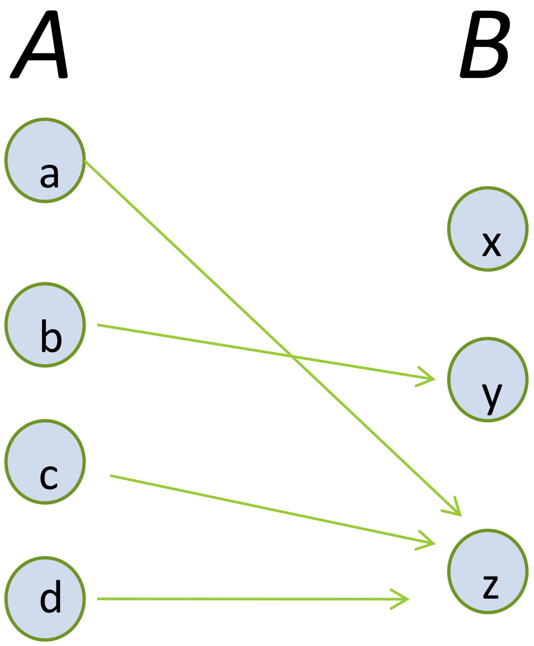
The domain of f is ? *A*

The codomain of f is ? *B*

The preimage of y is ? b

f(*A*) = ? {y, z}

The preimage(s) of z is (are) ? {a,c,d}



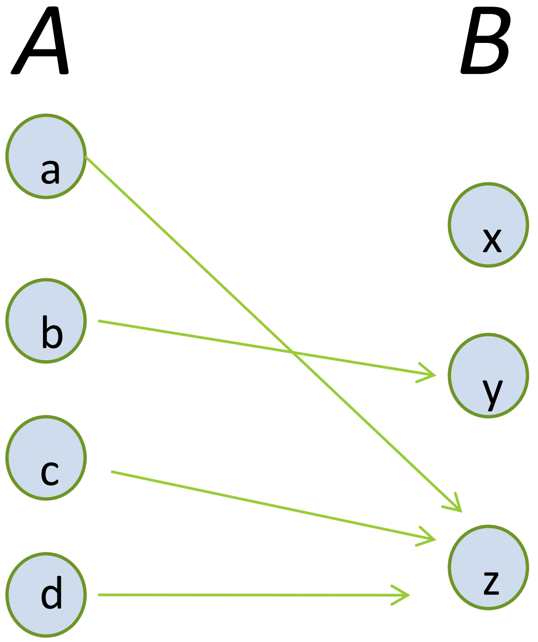
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}

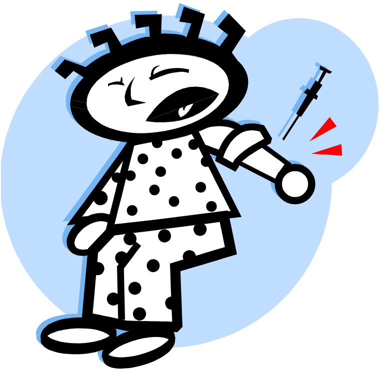


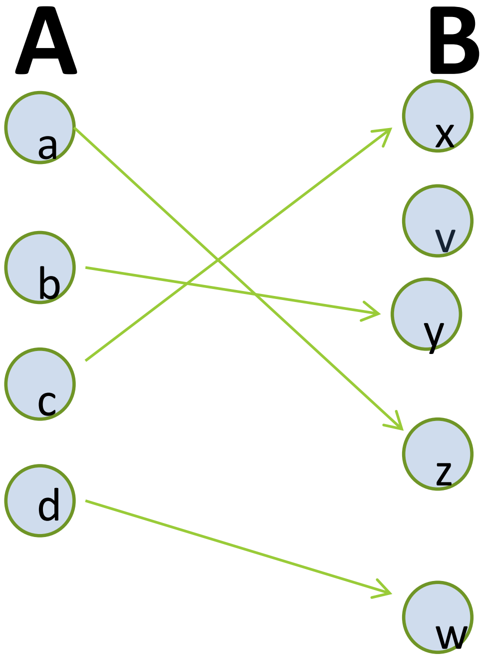
Injections

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if *f*(*a*) = *f*(*b*) implies that *a* = *b* for all *a* and *b* in the domain of *f*.

A function is said to be **an *injection* if it is one-to-one** (**preimages are unique**).

- Note: this means that if a ≠ b then f(a) ≠ f(b).



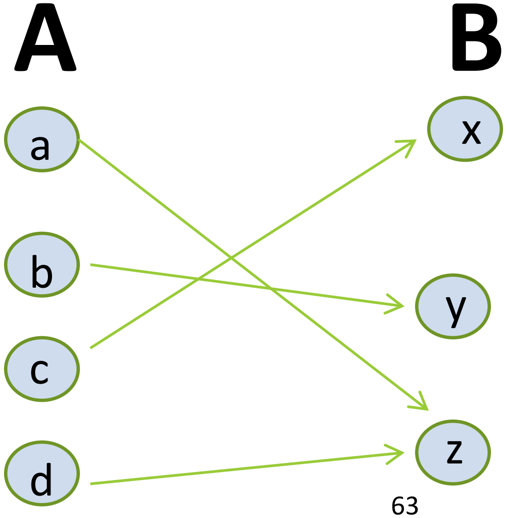


Surjections

Definition: A function *f* from *A* to *B* is called ***onto*** or ***surjective***, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

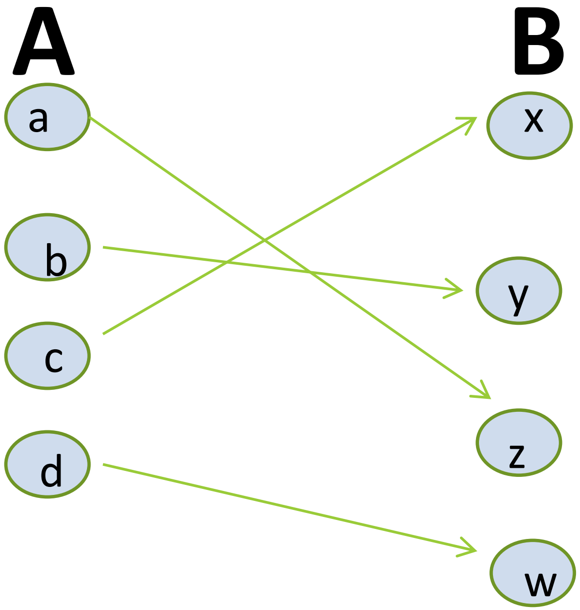
A function *f* is called a *surjection* if it is onto (**if every y in B has a preimage**).

Note: this means that for every y in B there must be an x in A such that f(x) = y.



Bijections

Definition: A function f is a ***one-to-one correspondence***, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



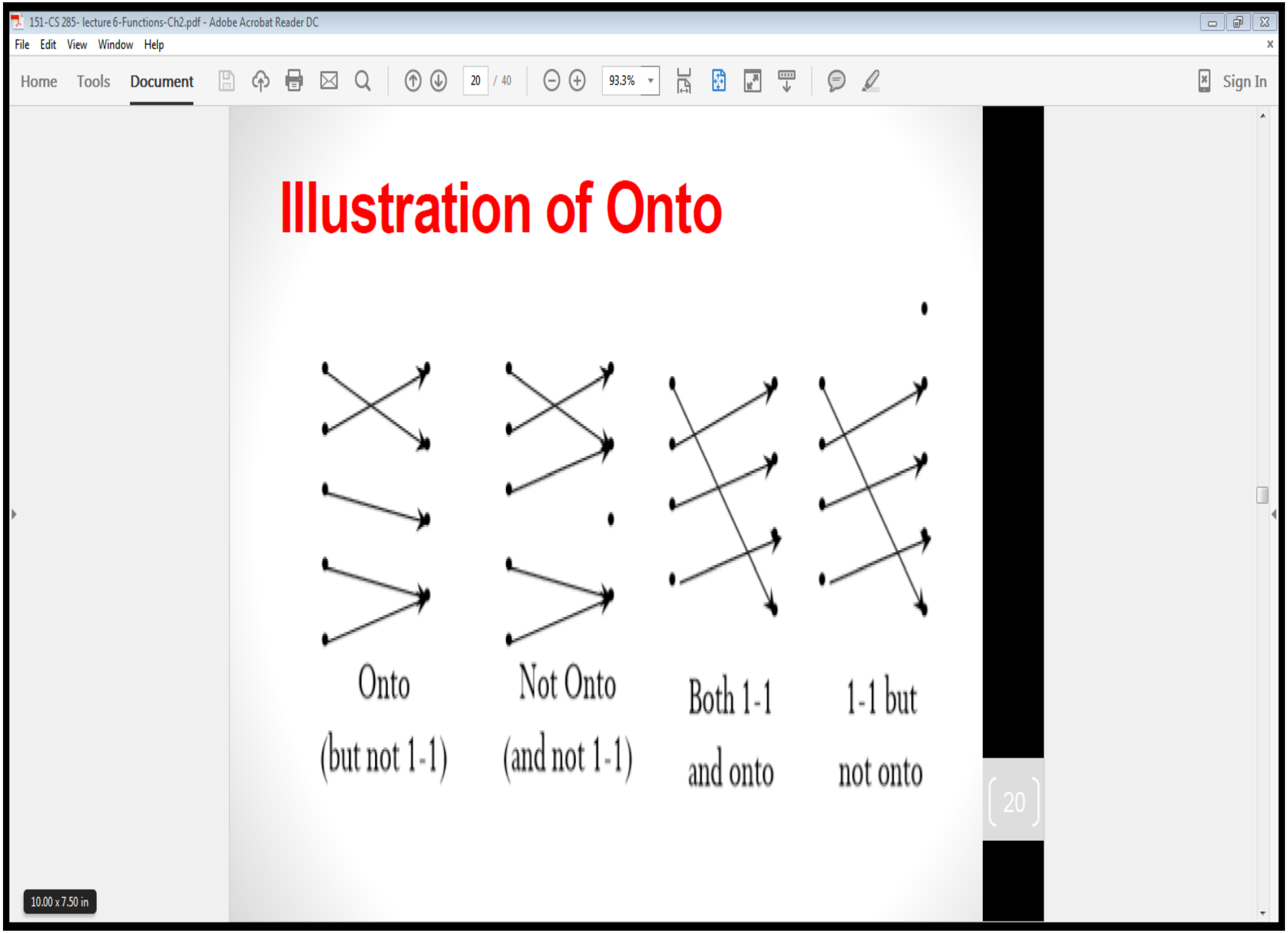


Illustration of Onto

Onto
(but not 1-1)

Not Onto
(and not 1-1)

Both 1-1
and onto

1-1 but
not onto

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by: $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain.

- *If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.*

Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Showing that *f* is one-to-one or onto

- **Example 3**: Let *f* be the function *f*(*x*) = *3x-2* from the set of R. Is *f* one-to-one function?
- Solution: Yes
- From definition, the function is one-to-one if and only if x1 ≠ x2 then f(x1) ≠ f(x2).

$$f(x_1)=f(x_2)$$

$$3\,x_1 - 2 = 3x_2 - 2$$

$$3\,x_1 = 3x_2$$

$$x_1 = x_2$$

•**g** is injective, since if b∈R, then g(x)=b has at most one solution

Injections, Surjections and Bijections

Let A = B = R, the reals. Determine which are injections, surjections, bijections:

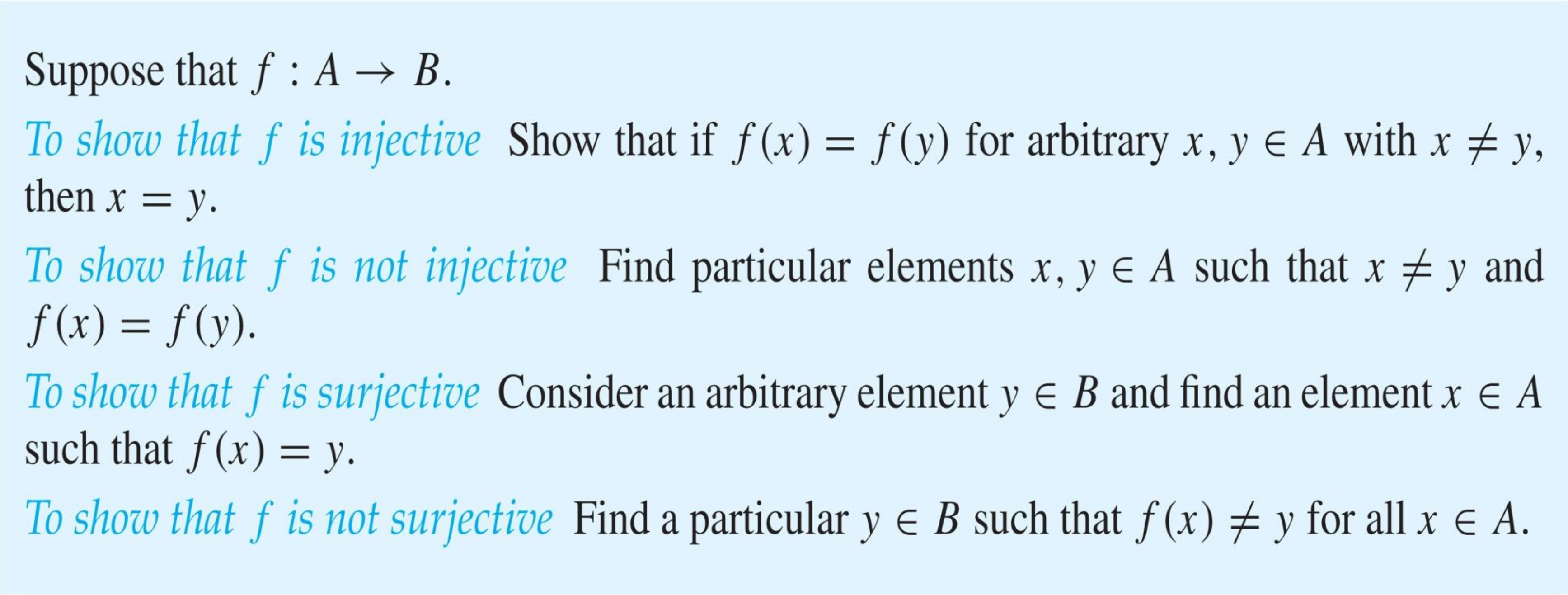
- $f(x) = x$,
- $f(x) = x^2$,
- $f(x) = x^3$,
- $f(x) = x + \sin(x)$,
- $f(x) = | x |$

Injections, Surjections and Bijections

Let A = B = R, the reals. Determine which are injections, surjections, bijections:

- f(x) = x, injective, surjective, bijective
- f(x) = x^2, none
- f(x) = x^3, injective, surjective, bijective
- f(x) = x + sin(x), injective, surjective, bijective
- f(x) = | x |, none

Showing that *f* is one-to-one or onto



Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Recall

- **Injection (One-to-One)**

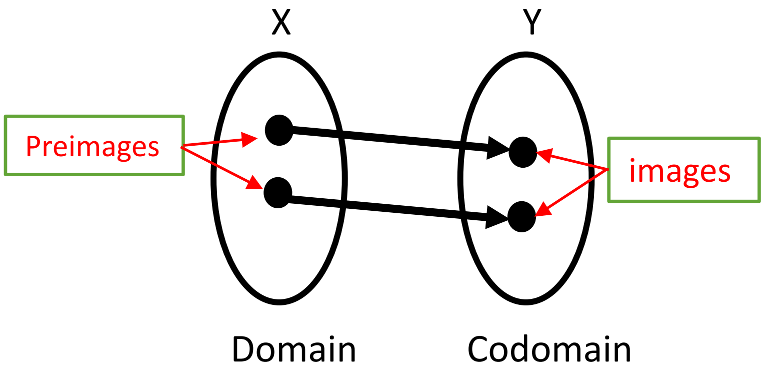
Preimages are unique

- **Surjection (Onto)**

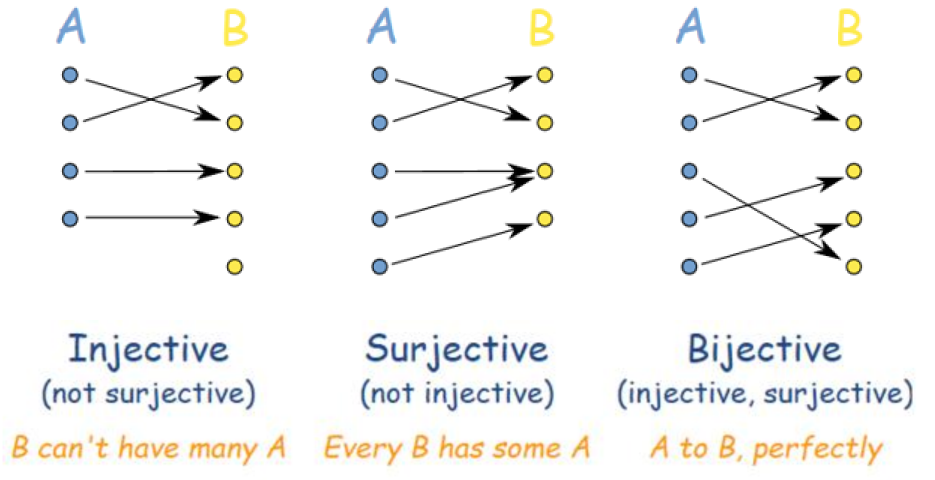
Every y has at least preimage

- **Bijection**

A function that is both an injection and a surjection.



Function types

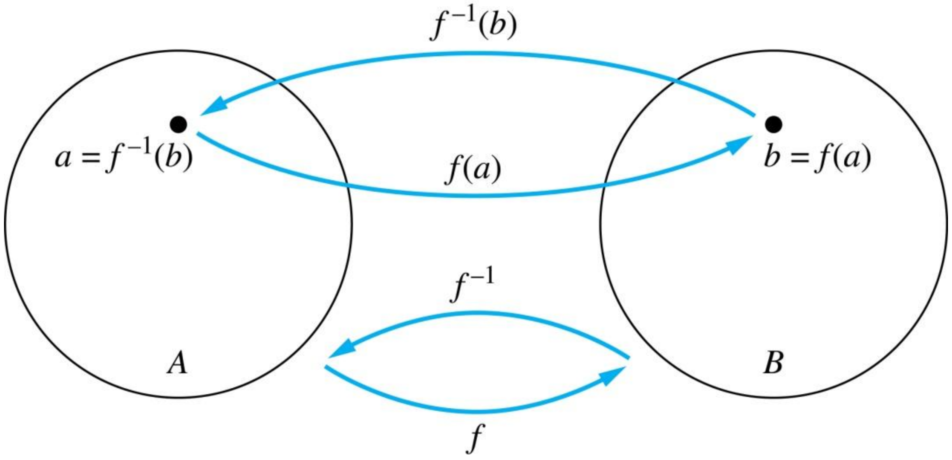


Inverse Functions

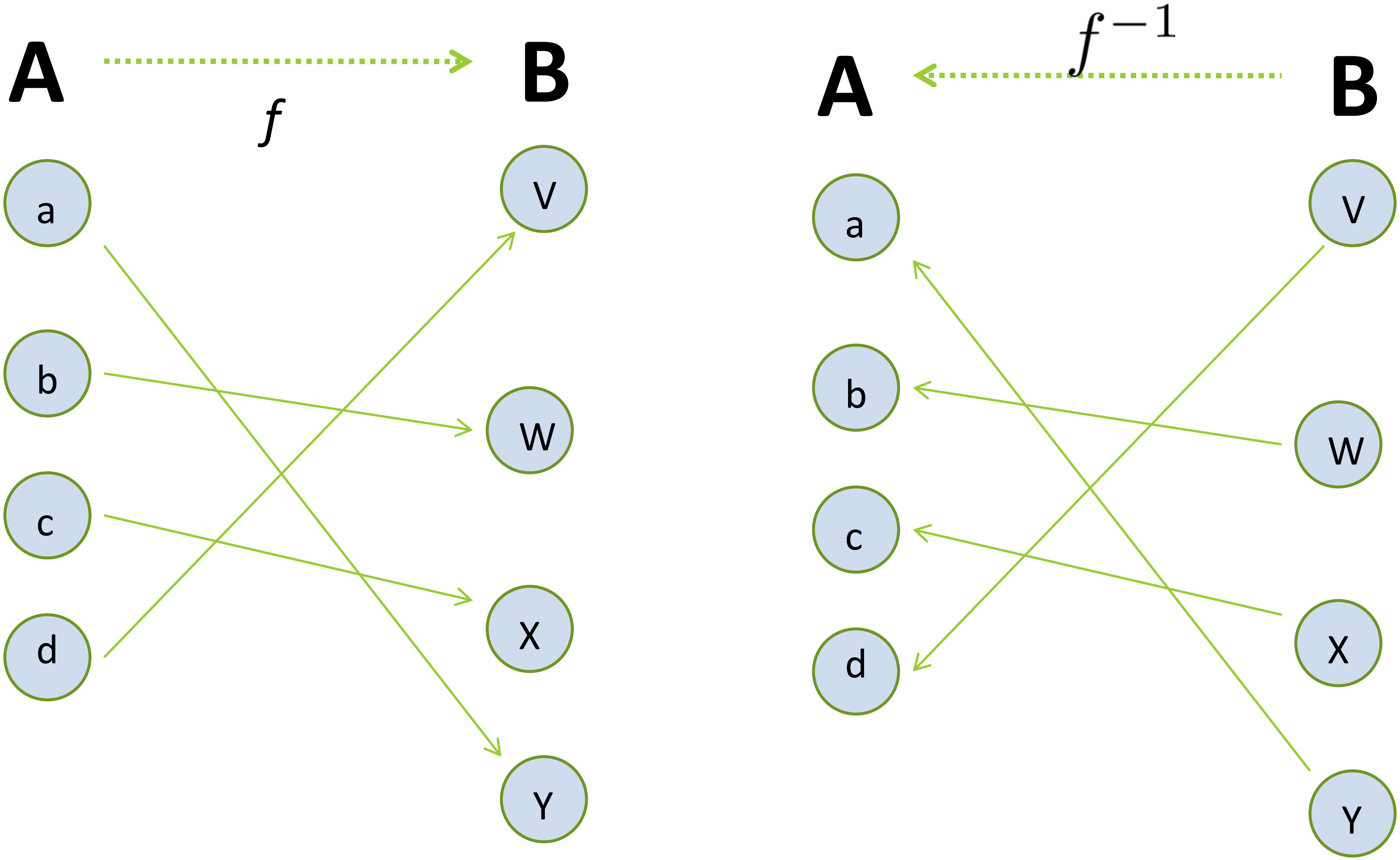
Definition: Let *f* be a **bijection** from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless *f* is a bijection. Why?



Inverse Functions



Questions

Example 1: Let *f* be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that

$f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.

Is f invertible and if so what is its inverse?

Solution:
The function f is invertible because it is a one-to-one correspondence. The inverse function reverses the correspondence given by f,
so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

Example 3: Let *f*: **R** → **R** be such that $f(x) = x^2$. Is *f* invertible, and if so, what is its inverse?

Solution:

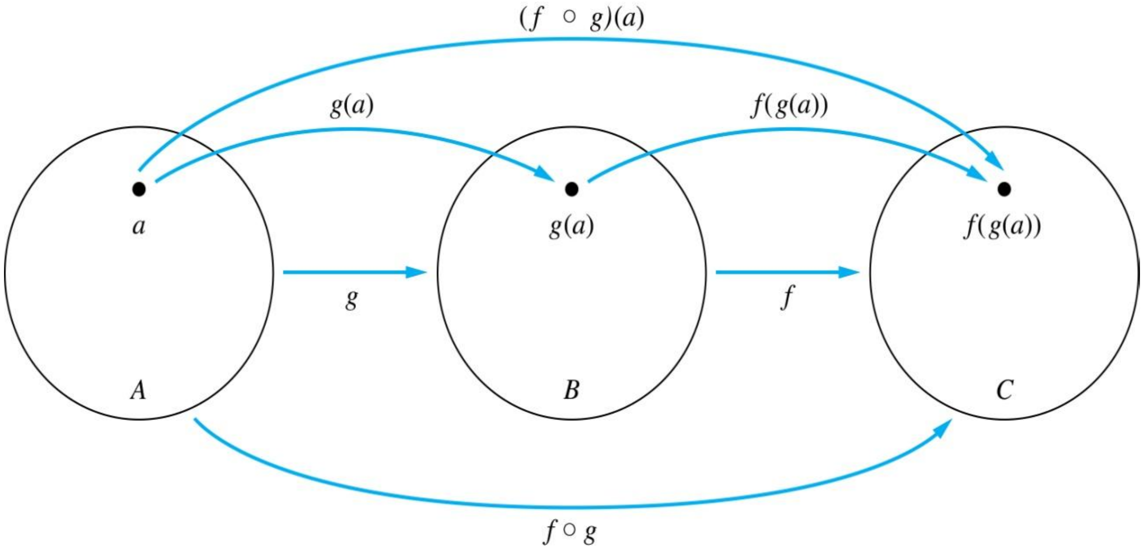
The function *f* is not invertible because it is not one-to-one .

Composition

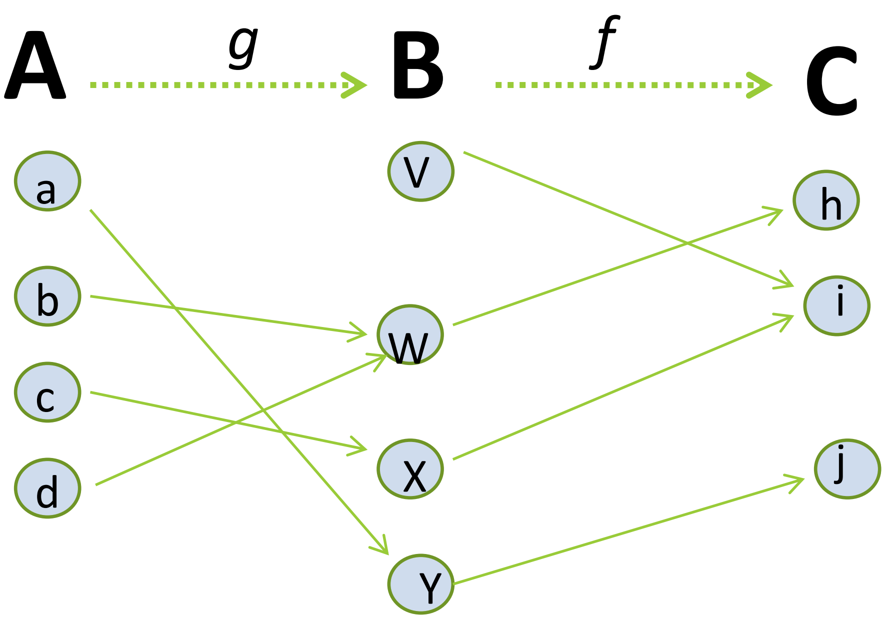
Composition

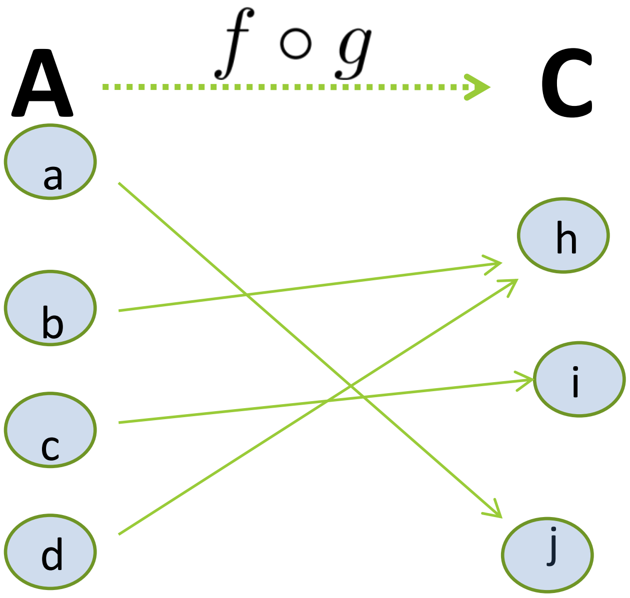
- **Definition**: Let *g*: *A* → *B, f*: *B* → *C*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by

$$f \circ g(x) = f(g(x))$$



Composition





Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

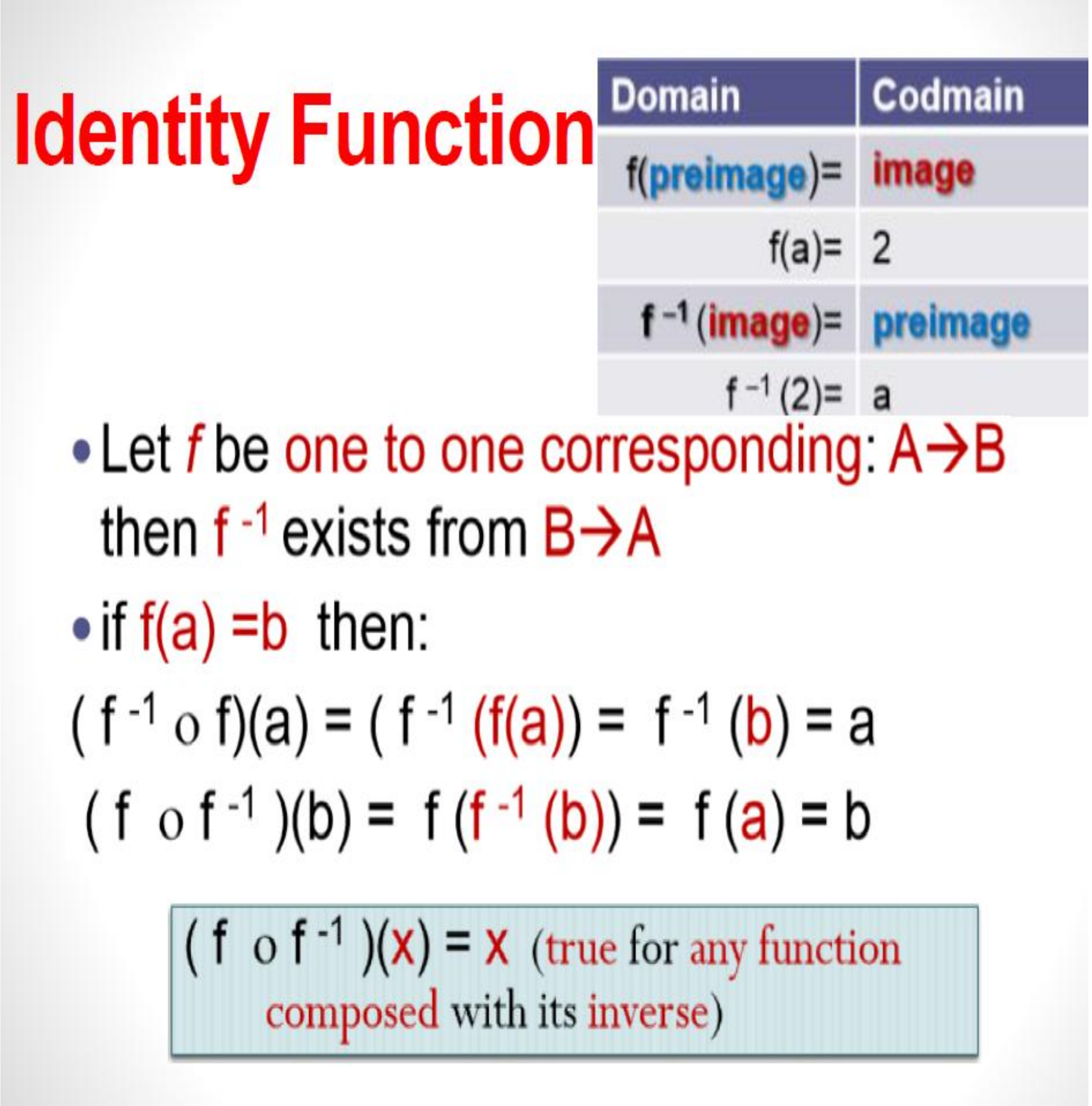
Example 3: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f ?

Solution:

$$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

$$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$



Identity Function

Domain	Codmain
f(preimage)=	image
f(a)=	2
f^{-1}(image)=	preimage
$f^{-1}(2)$=	a

- Let *f* be one to one corresponding: A→B then f^{-1} exists from B→A
- if $f(a) = b$ then:

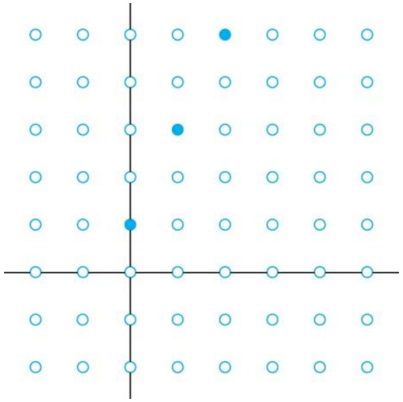
$$(f^{-1} \circ f)(a) = (f^{-1}(f(a)) = f^{-1}(b) = a$$

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$$

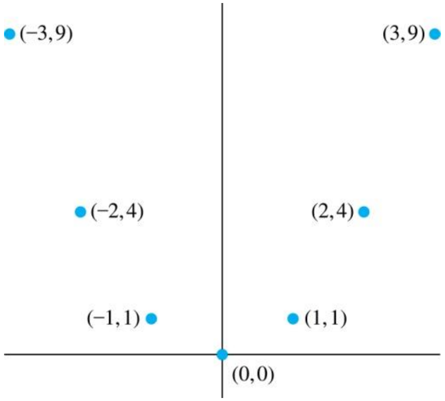
$(f \circ f^{-1})(x) = x$ (true for any function composed with its inverse)

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$
from Z to Z



Graph of $f(x) = x^2$
from Z to Z

Floor and Ceiling Functions

Some Important Functions

- The ***floor*** function, denoted $f(x) = \lfloor x \rfloor$

  is the largest integer *less than or equal* to *x*.

- The ***ceiling*** function, denoted $f(x) = \lceil x \rceil$

  is the smallest integer *greater than or equal* to *x*

Some Important Functions

Example:

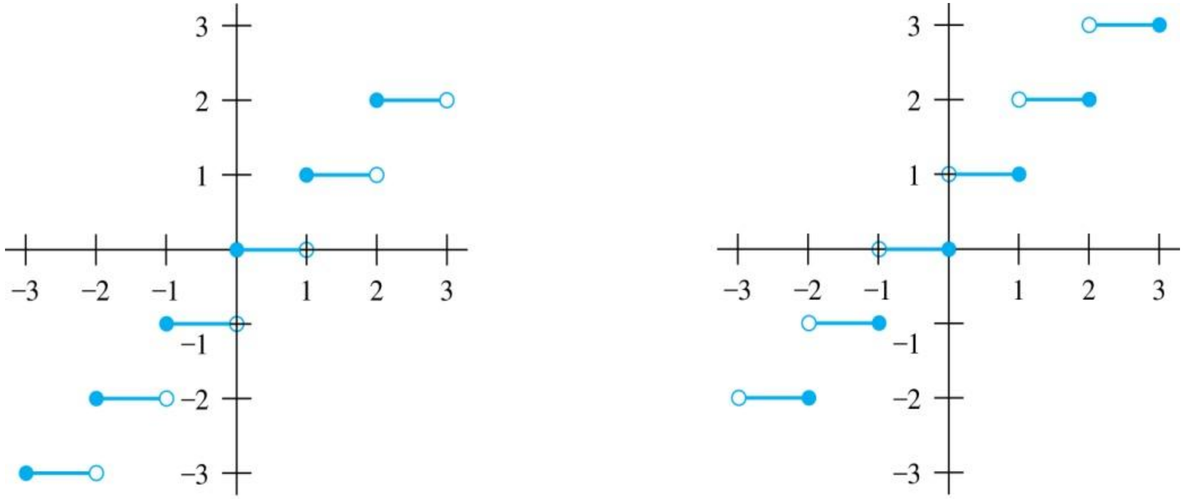
$$\lceil 3.5 \rceil = 4$$

$$\lfloor 3.5 \rfloor = 3$$

$$\lceil -1.5 \rceil = -1$$

$$\lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions



(a) $y = [x]$ (b) $y = [x]$

Graph of (a) Floor and (b) Ceiling Functions

Factorial Function

Definition: f: **N → Z⁺ ,** denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

Examples:

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Sequences and Summations

SECTION 2.4

Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
- Special Integer Sequences (*optional*)

Introduction

- **Sequences** are ordered lists of elements.
    - 1, 2, 3, 5, 8
    - 1, 3, 9, 27, 81, …….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, \ldots..\}$ or $\{1, 2, 3, 4, \ldots.\}$) to a set *S*.

- The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of *f(n)* where *f* is a function from $\{0,1,2,\ldots..\}$ to *S*. We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are **real numbers**.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

Definition: A *arithmetic progression* is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Special Integer Sequences (*opt*)

- Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.
- Some questions to ask?
  - Are there repeated terms of the same value?
  - Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
  - Can you obtain a term by combining the previous terms in some way?
  - Are they cycles among the terms?
  - Do the terms match those of a well known sequence?

Questions on Special Integer Sequences (*opt*)

Example 1: Find formulae for the sequences with the following first five terms: 1, ½, ¼, 1/8, 1/16

Solution: Note that the denominators are powers of 2. The sequence with $a_n = 1/2^n$ is a **possible** match. This is a geometric progression with $a = 1$ and $r = ½$.

Example 2: Consider 1,3,5,7,9

Solution: Note that each term is obtained by adding 2 to the previous term. A **possible** formula is $a_n = 2n + 1.$ This is an arithmetic progression with $a = 1$ and $d = 2$.

Example 3: 1, -1, 1, -1,1

Solution: The terms alternate between 1 and -1. A **possible** sequence is $a_n = (-1)^n$. This is a geometric progression with $a = 1$ and $r = -1$.

Useful Sequences

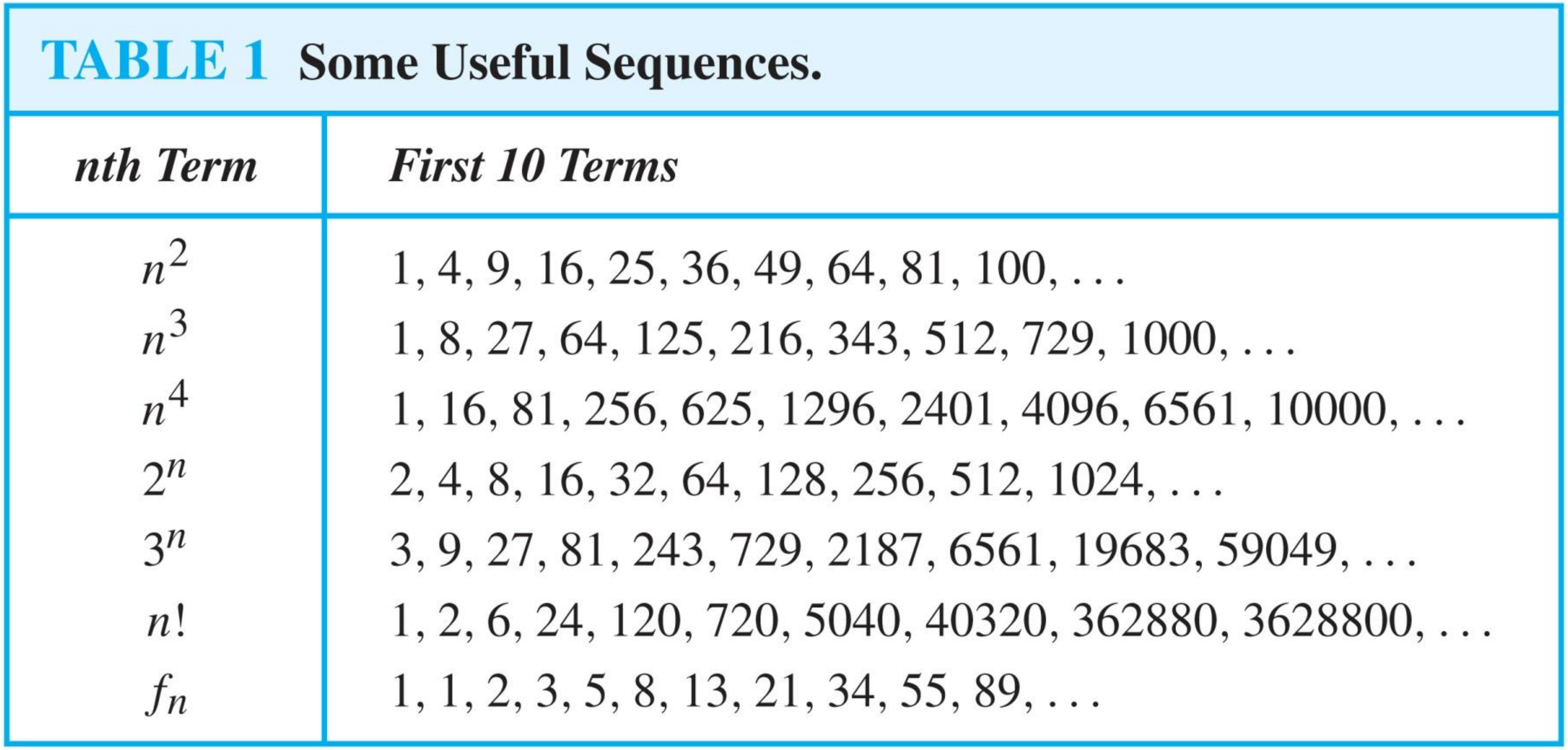


TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Guessing Sequences (*optional*)

Example: Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are $1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047$.

Solution: Note the ratio of each term to the previous approximates 3. So now compare with the sequence 3^n. We notice that the nth term is 2 less than the corresponding power of 3. So a good conjecture is that $a_n = 3^n - 2$.

Integer Sequences (*optional*)

- Integer sequences appear in a wide range of contexts. Later we will see the sequence of prime numbers (Chapter 4), the number of ways to order *n* discrete objects (Chapter 6), the number of moves needed to solve the Tower of Hanoi puzzle with *n* disks (Chapter 8), and the number of rabbits on an island after *n* months (Chapter 8).
- Integer sequences are useful in many fields such as biology, engineering, chemistry and physics.
- On-Line Encyclopedia of Integer Sequences (OESIS) contains over 200,000 sequences. Began by Neil Stone in the 1960s (printed form). Now found at http://oeis.org/Spuzzle.html

Integer Sequences (*optional*)

- Here are three interesting sequences to try from the OESIS site. To solve each puzzle, find a rule that determines the terms of the sequence.
- Guess the rules for forming for the following sequences:
  - 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, ...
    - Hint: Think of adding and multiplying by numbers to generate this sequence.
  - 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, ...
    - Hint: Think of the English names for the numbers representing the position in the sequence and the Roman Numerals for the same number.
  - 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, ...
    - Hint: Think of the English names for numbers, and whether or not they have the letter ‘e.’
- The answers and many more can be found at

http://oeis.org/Spuzzle.html

Summations

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$

  from the sequence $\{a_n\}$
- The notation:

$$\sum_{j=m}^{n} a_j \quad \textstyle\sum_{j=m}^{n} a_j \quad \sum_{m \le j \le n} a_j$$

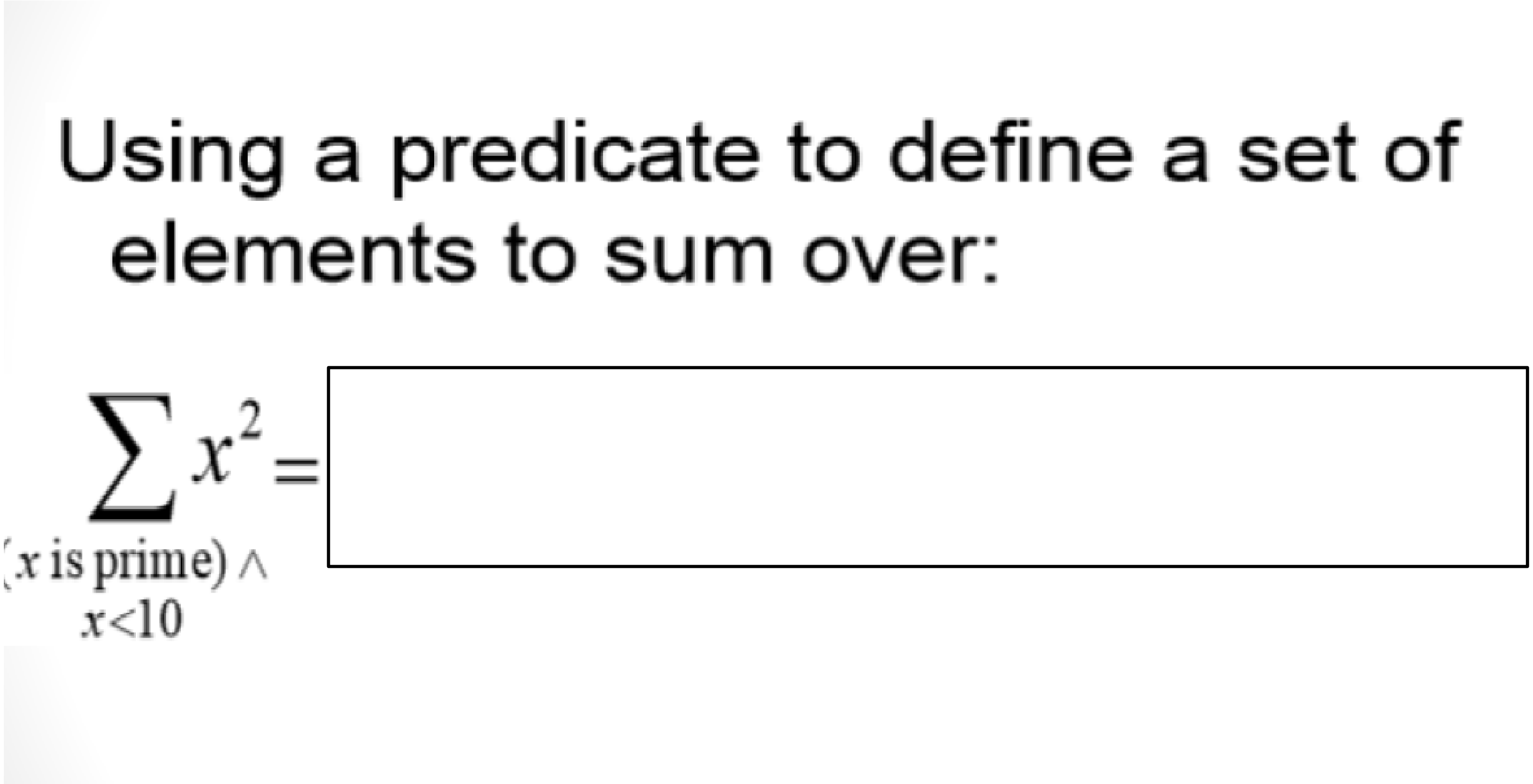
represents $a_m + a_{m+1} + \cdots + a_n$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit* ***m*** and ending with its *upper limit* ***n***.

Example

$$\sum_{i=2}^{4}(i^2+1)$$

Example



Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \wedge \\ x<10}} x^2 = \boxed{\qquad\qquad\qquad\qquad}$$

Summations

- More generally for a set *S*:

$$\textstyle\sum_{j\in S} a_j$$

- **Examples**:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j\in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation (*optional*)

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$

  from the sequence $\{a_n\}$

- The notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{m\leq j\leq n} a_j$$

  represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Some Useful Summation Formulae

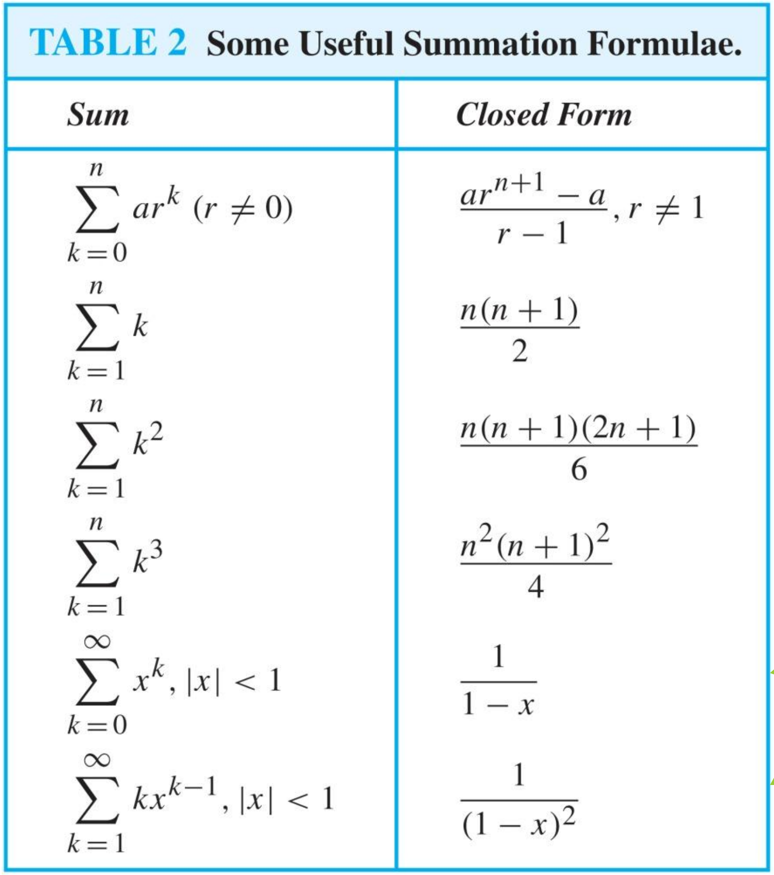
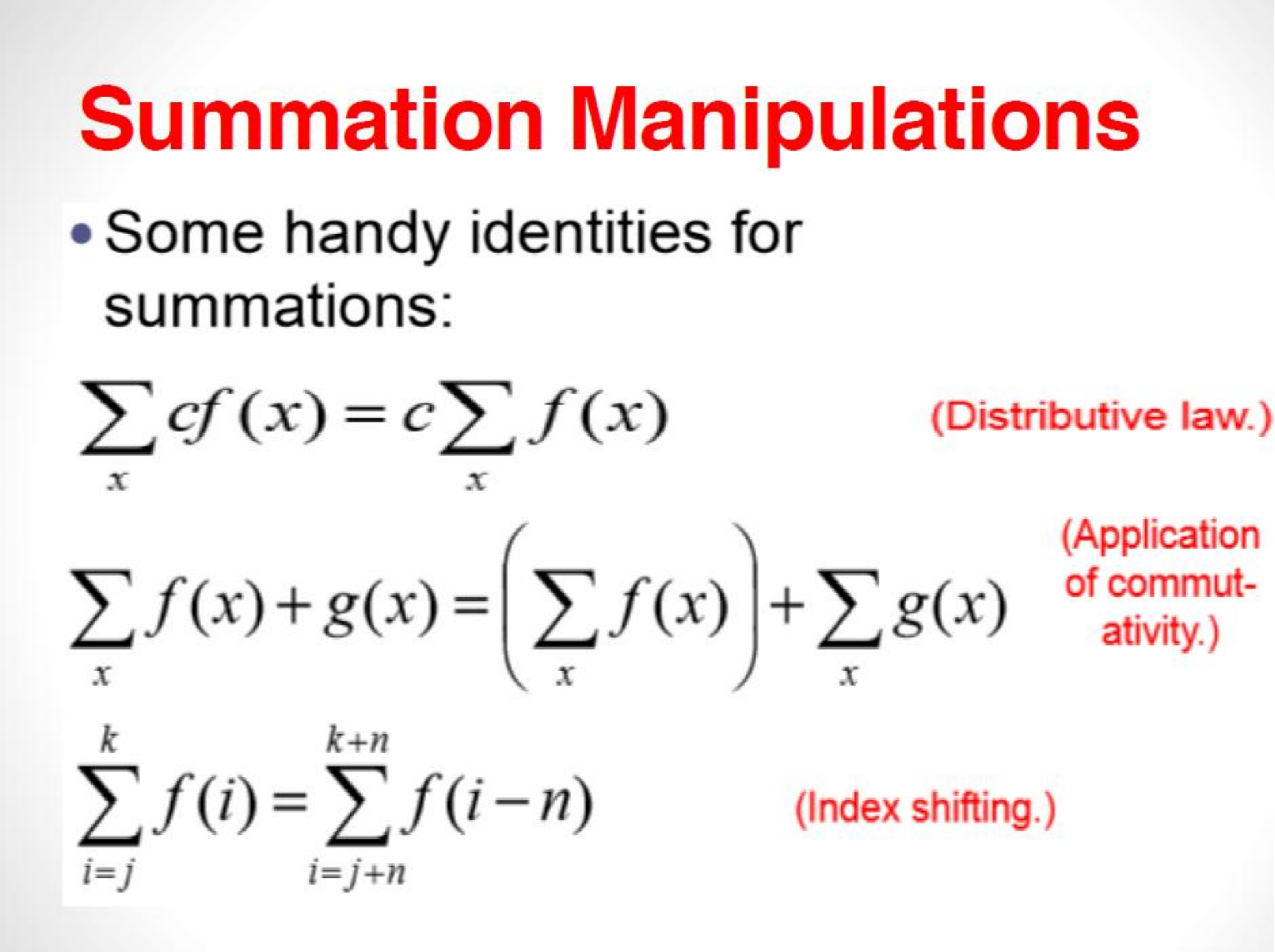


TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Later we will prove some of these by induction.

Proof in text (requires calculus)



Summation Manipulations

- Some handy identities for summations:

$$\sum_x cf(x) = c\sum_x f(x)$$

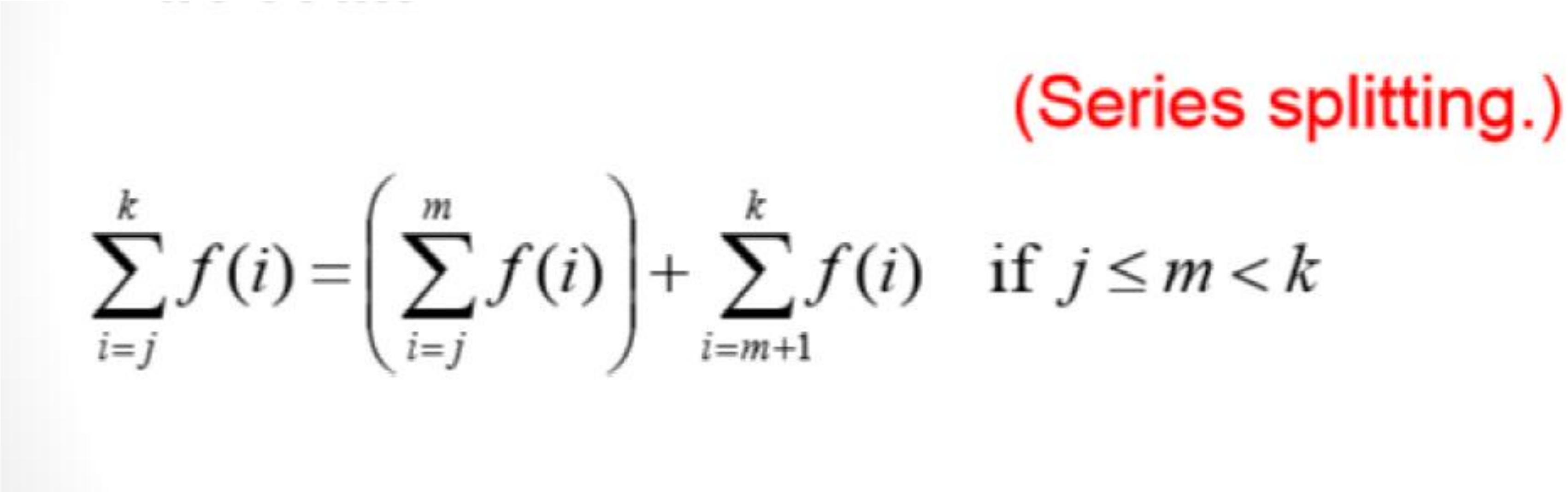
(Distributive law.)

$$\sum_x f(x) + g(x) = \left(\sum_x f(x)\right) + \sum_x g(x)$$

(Application of commutativity.)

$$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n)$$

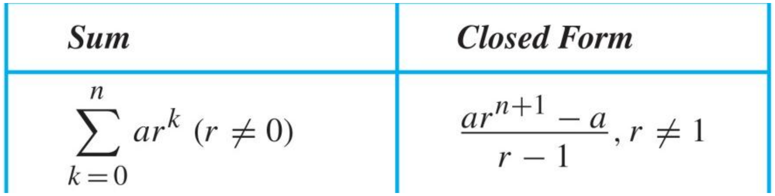
(Index shifting.)



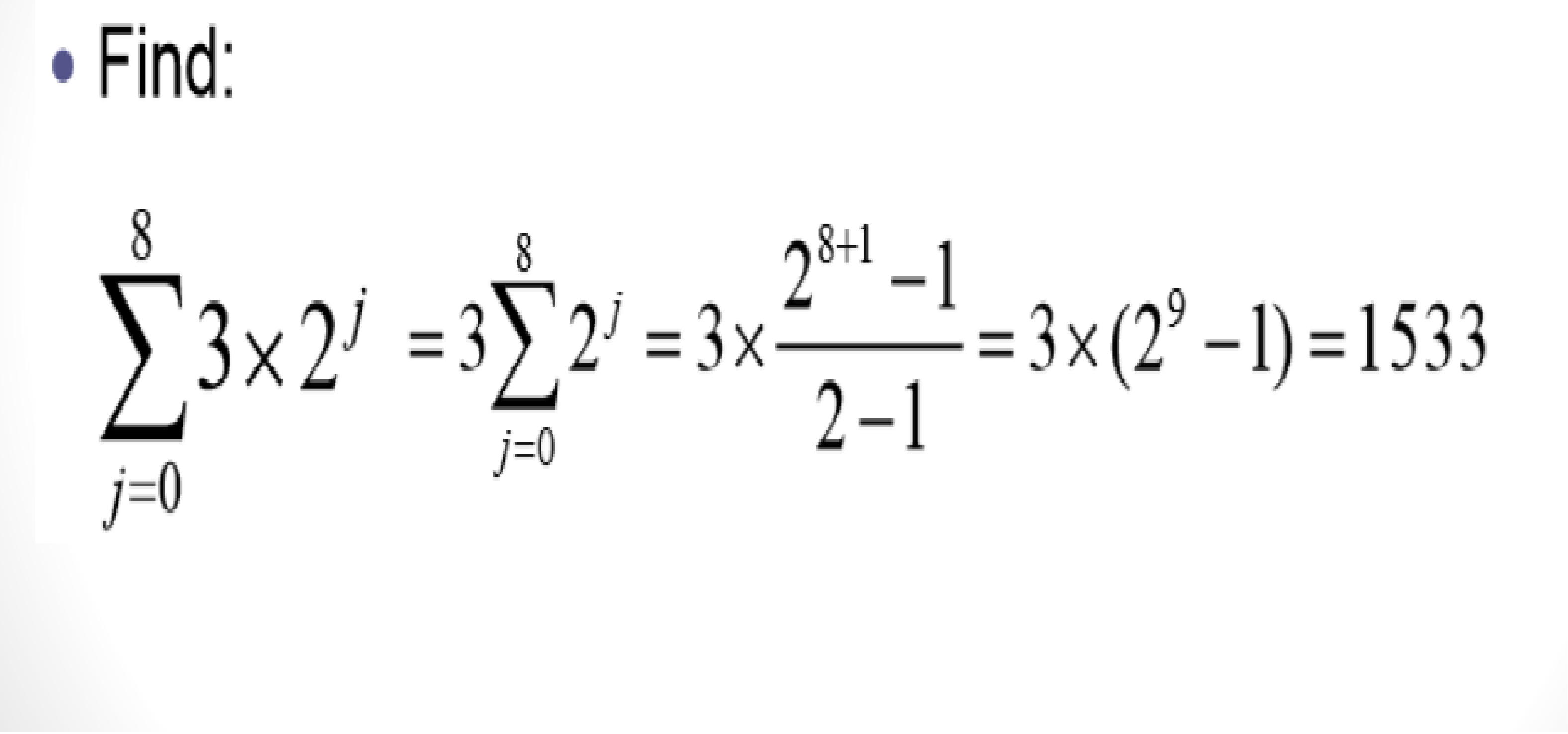
(Series splitting.)

$$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i) \right) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

Example



Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$



- Find:

$$\sum_{j=0}^{8} 3 \times 2^j = 3\sum_{j=0}^{8} 2^j = 3 \times \frac{2^{8+1} - 1}{2 - 1} = 3 \times (2^9 - 1) = 1533$$

Using the Shortcuts

$$\sum_{k=50}^{100} k^2$$

- Example: Evaluate
  - Use series splitting.
  - Solve for desired summation.
  - Apply quadratic series rule.
  - Evaluate.

$$\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$$

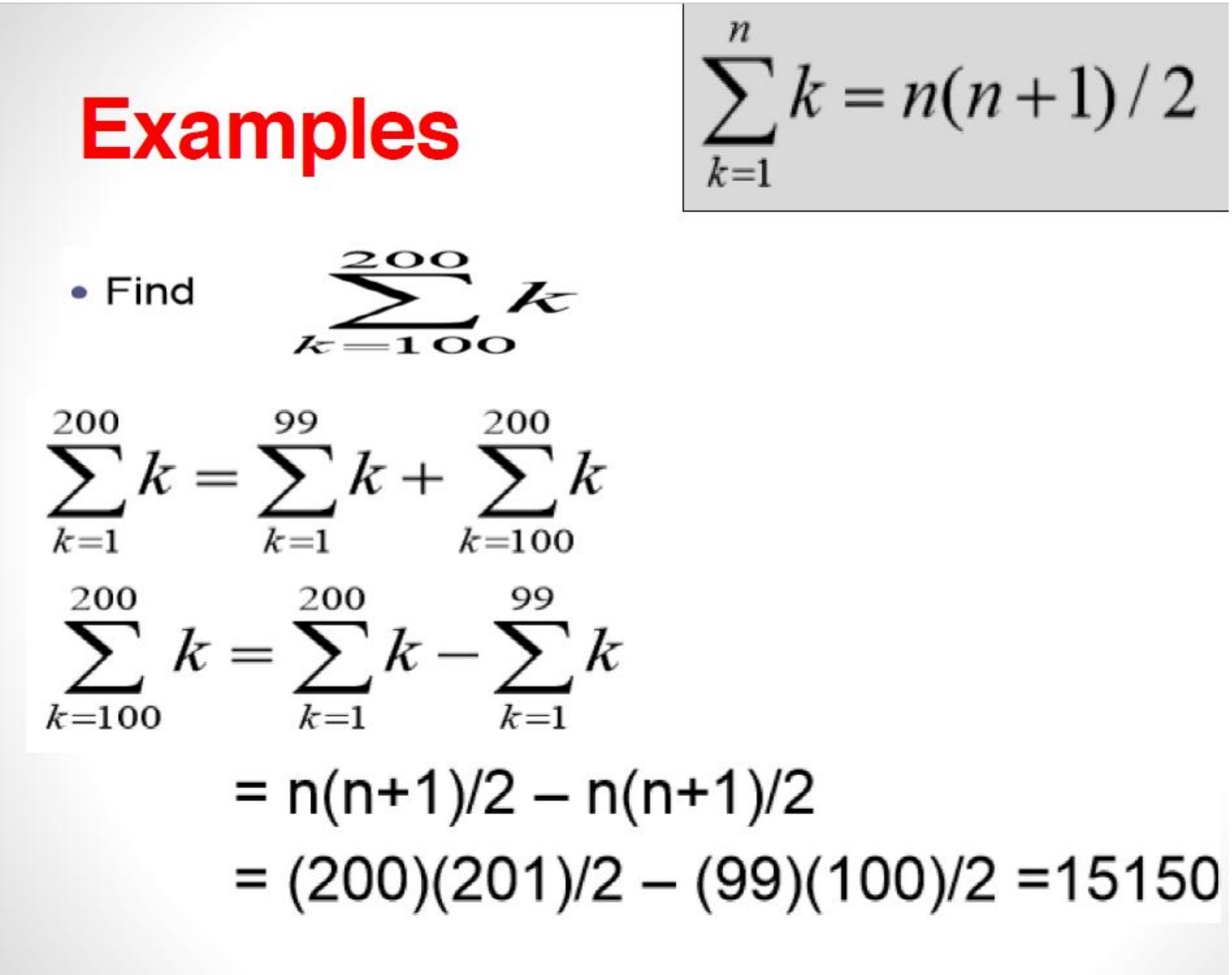
$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338{,}350 - 40{,}425$$

$$= 297{,}925.$$

$$\boxed{\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6}$$

from Table 2, slide 113



Examples

$$\sum_{k=1}^{n} k = n(n+1)/2$$

- Find $\sum_{k=100}^{200} k$

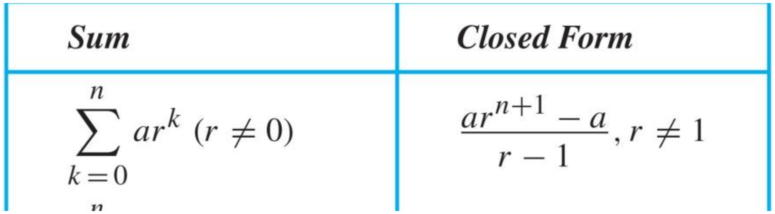
$$\sum_{k=1}^{200} k = \sum_{k=1}^{99} k + \sum_{k=100}^{200} k$$

$$\sum_{k=100}^{200} k = \sum_{k=1}^{200} k - \sum_{k=1}^{99} k$$

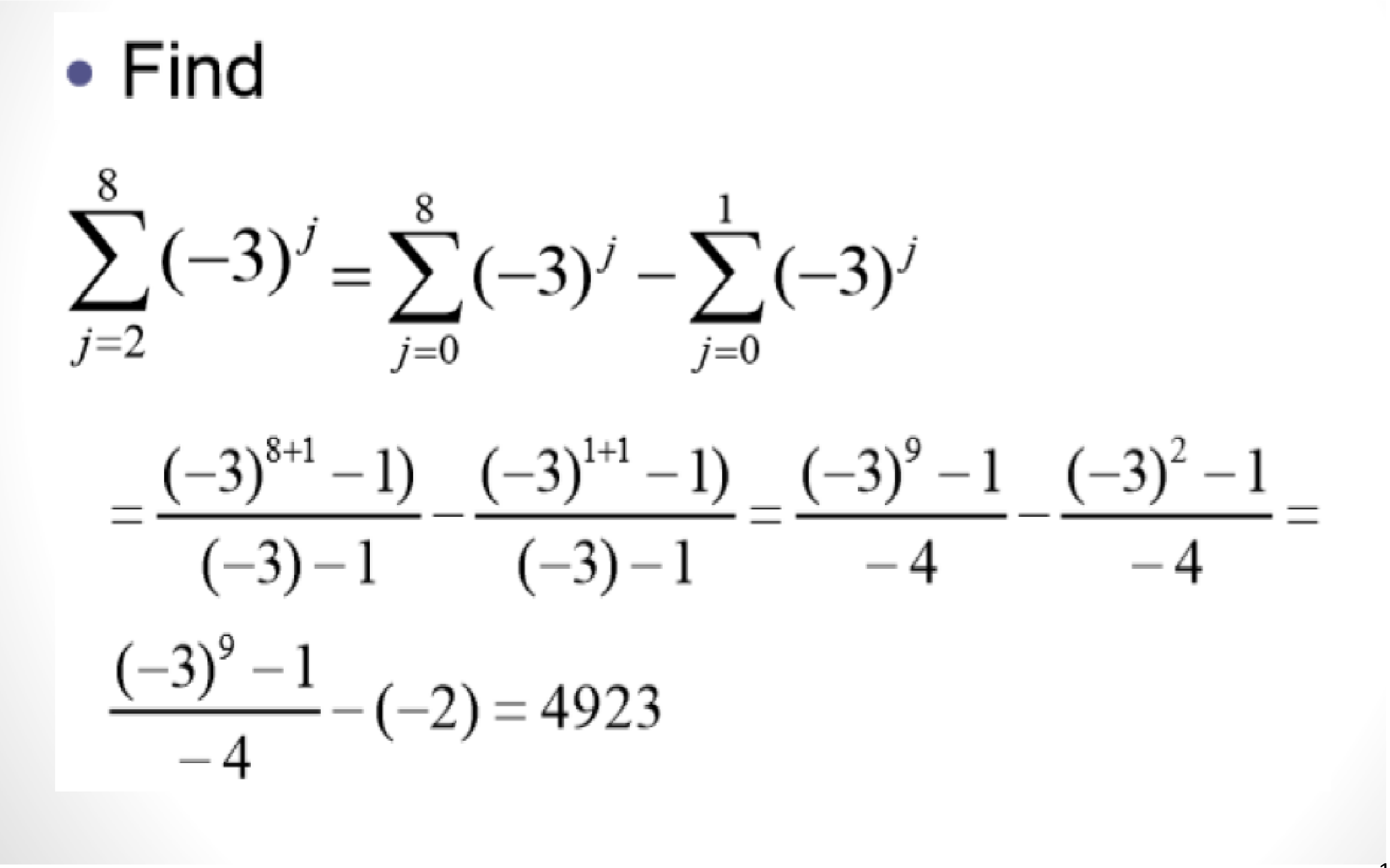
= n(n+1)/2 – n(n+1)/2

= (200)(201)/2 – (99)(100)/2 =15150

Example



Sum	Closed Form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$



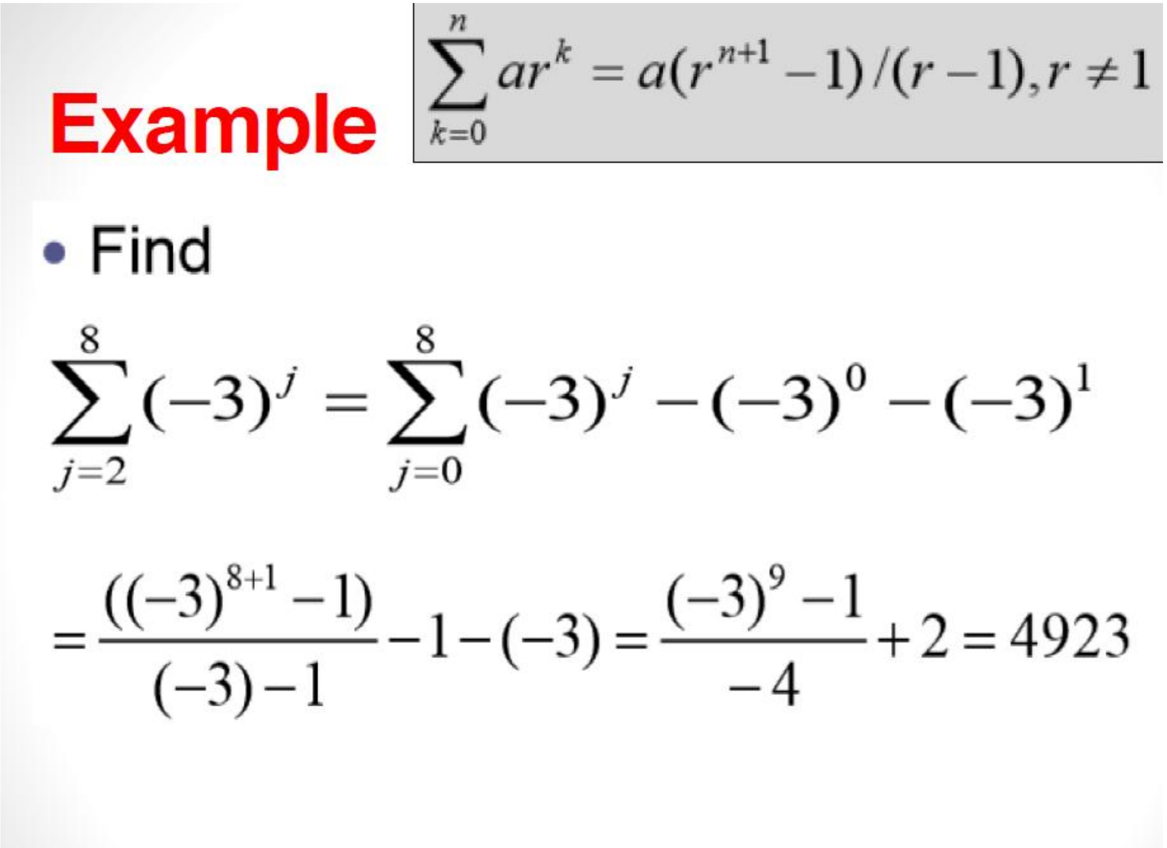
- Find

$$\sum_{j=2}^{8}(-3)^j = \sum_{j=0}^{8}(-3)^j - \sum_{j=0}^{1}(-3)^j$$

$$= \frac{(-3)^{8+1}-1)}{(-3)-1} - \frac{(-3)^{1+1}-1)}{(-3)-1} = \frac{(-3)^9-1}{-4} - \frac{(-3)^2-1}{-4} =$$

$$\frac{(-3)^9-1}{-4} - (-2) = 4923$$

Example



Example

$$\sum_{k=0}^{n} ar^k = a(r^{n+1}-1)/(r-1), r \neq 1$$

- Find

$$\sum_{j=2}^{8}(-3)^j = \sum_{j=0}^{8}(-3)^j - (-3)^0 - (-3)^1$$

$$= \frac{((-3)^{8+1}-1)}{(-3)-1} - 1 - (-3) = \frac{(-3)^9-1}{-4} + 2 = 4923$$

Recurrence Relations

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$\begin{aligned} a_1 &= a_0 + 3 = 2 + 3 = 5 \\ a_2 &= 5 + 3 = 8 \\ a_3 &= 8 + 3 = 11 \end{aligned}$$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

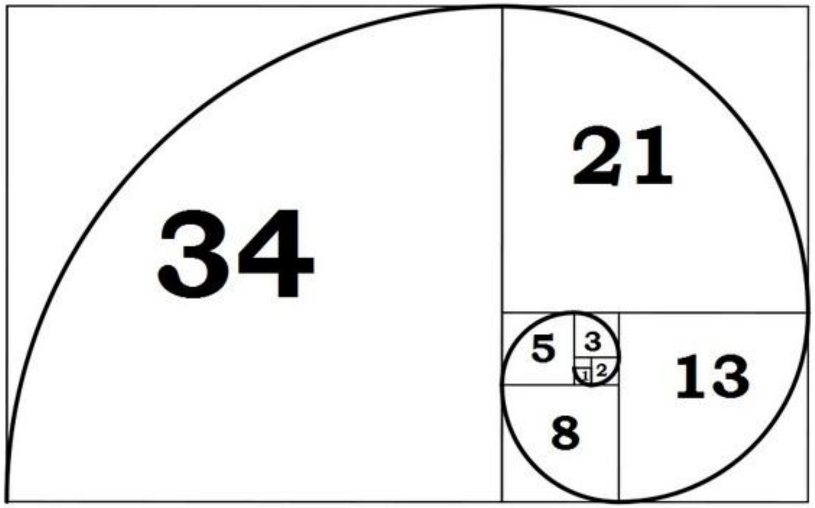
$f_2 = f_1 + f_0 = 1 + 0 = 1,$

$f_3 = f_2 + f_1 = 1 + 1 = 2,$

$f_4 = f_3 + f_2 = 2 + 1 = 3,$

$f_5 = f_4 + f_3 = 3 + 2 = 5,$

$f_6 = f_5 + f_4 = 5 + 3 = 8.$



Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a ***closed formula***.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Iterative Solution Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation:

$a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

.

.

.

$$a_n = a_{n\text{-}1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\ &= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\ &\;\;\vdots \\ &= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1) \end{aligned}$$

Financial Application

Example: Suppose that a person deposits $\$10{,}000.00$ in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\,P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Continued on next slide →

Financial Application

$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

:

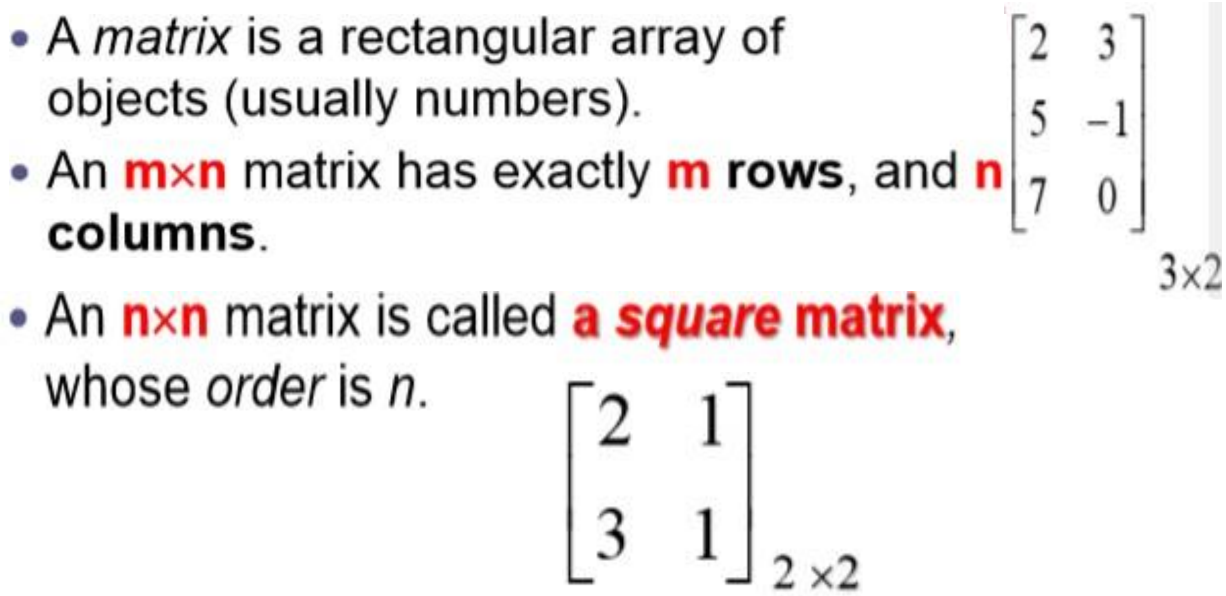
$P_n = (1.11)P_{n-1} = (1.11)^n\mathrm{P}_0 = (1.11)^n\, 10{,}000$

$P_n = (1.11)^n\, 10{,}000$ (Can prove by induction, covered in Chapter 5)

$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

2.6. Matrices

2.6. Matrices



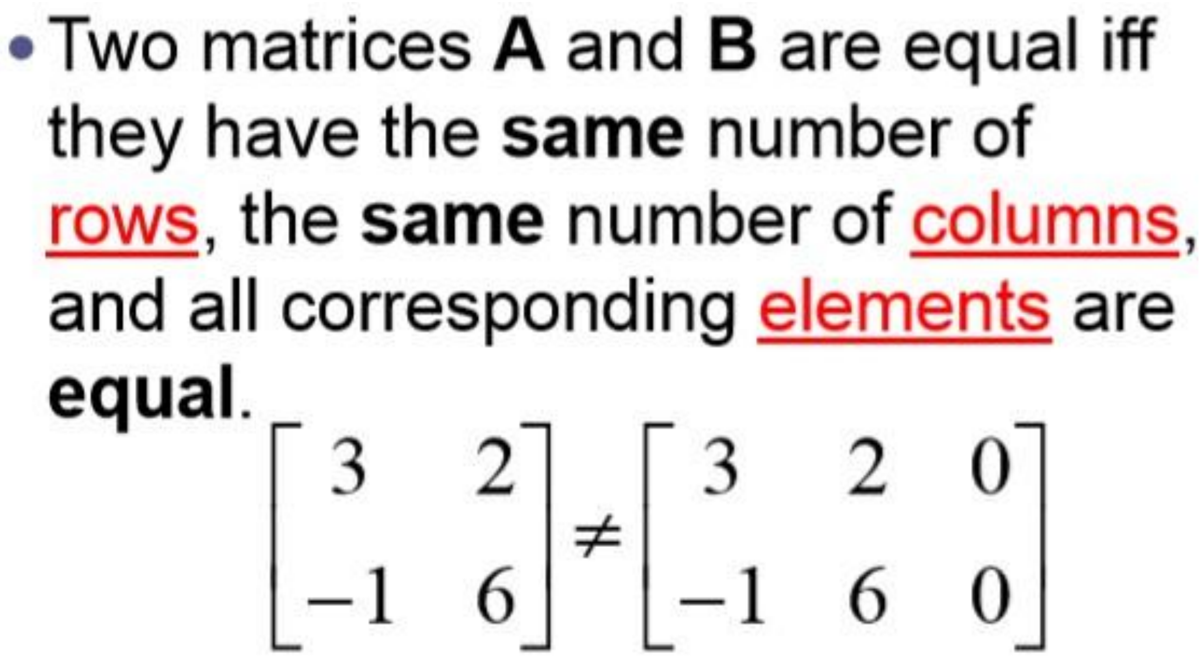
- A *matrix* is a rectangular array of objects (usually numbers).
- An **m×n** matrix has exactly **m rows**, and **n columns**.

$$\begin{bmatrix} 2 & 3 \\ 5 & -1 \\ 7 & 0 \end{bmatrix}_{3\times 2}$$

- An **n×n** matrix is called ***a square matrix***, whose *order* is *n*.

$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}_{2\times 2}$$

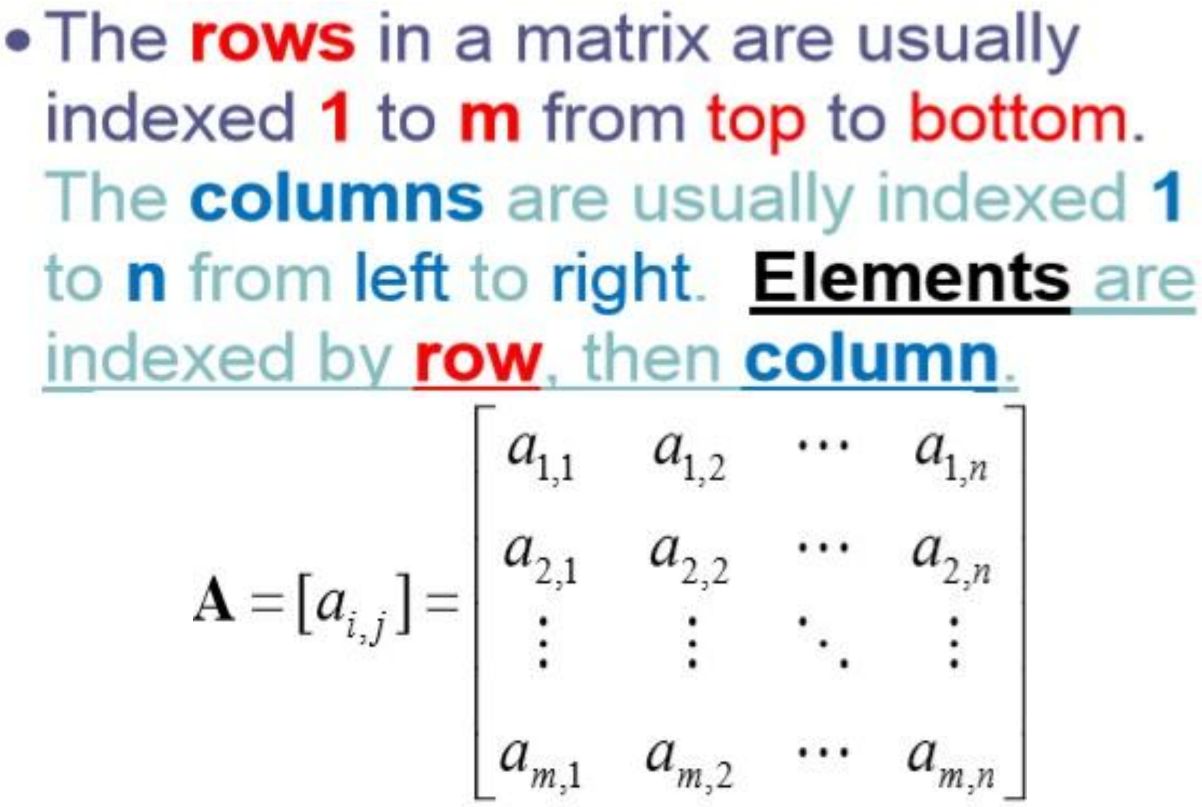
Matrix Equality



- Two matrices **A** and **B** are equal iff they have the **same** number of rows, the **same** number of columns, and all corresponding elements are **equal**.

$$\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 & 0 \\ -1 & 6 & 0 \end{bmatrix}$$

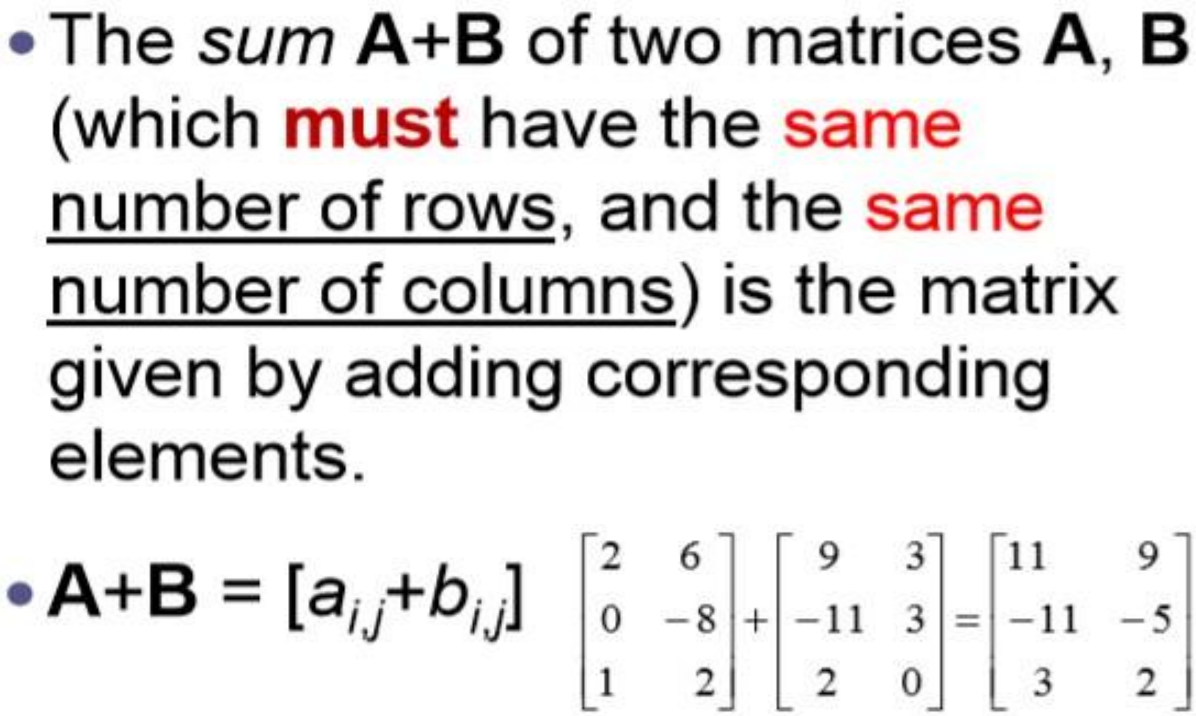
Row and Column Order



- The **rows** in a matrix are usually indexed **1** to **m** from top to bottom. The **columns** are usually indexed **1** to **n** from left to right. **Elements** are indexed by **row**, then **column**.

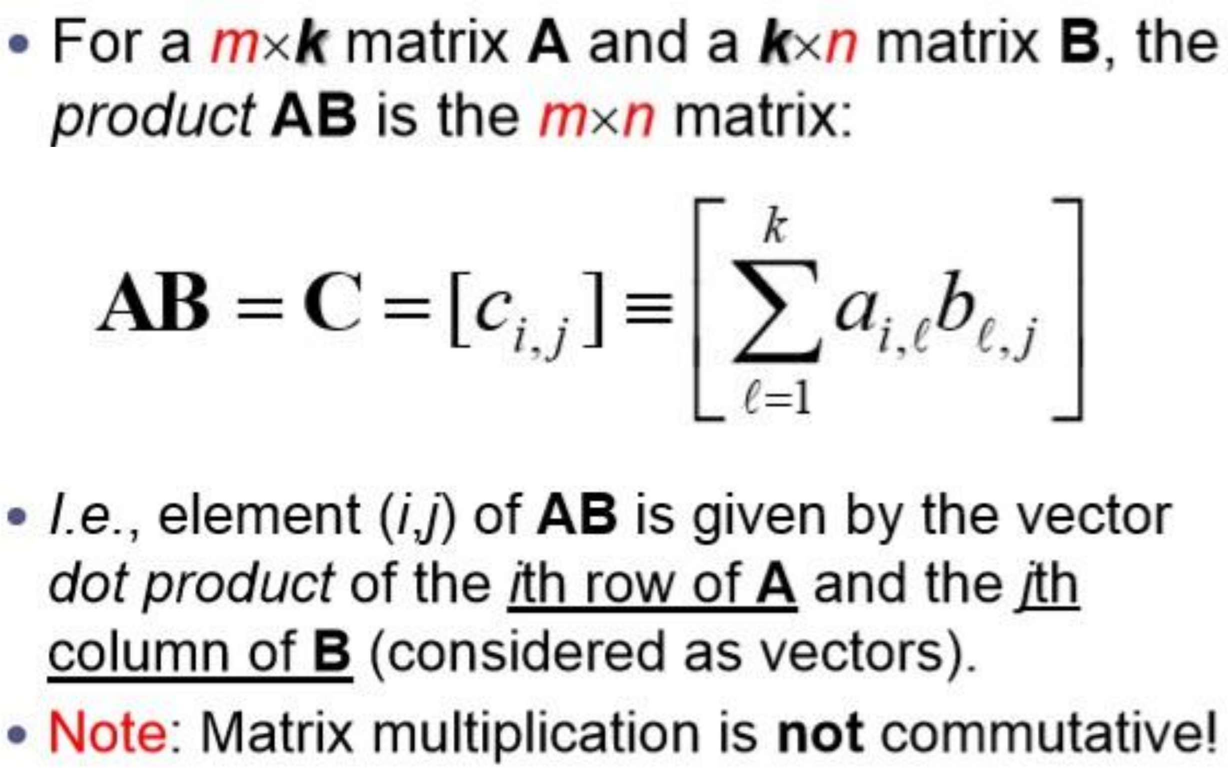
$$\mathbf{A} = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Matrix Sums



- The *sum* **A+B** of two matrices **A**, **B** (which **must** have the same number of rows, and the same number of columns) is the matrix given by adding corresponding elements.
- **A+B** = $[a_{i,j}+b_{i,j}]$ $\begin{bmatrix} 2 & 6 \\ 0 & -8 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ -11 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ -11 & -5 \\ 3 & 2 \end{bmatrix}$

Matrix Products



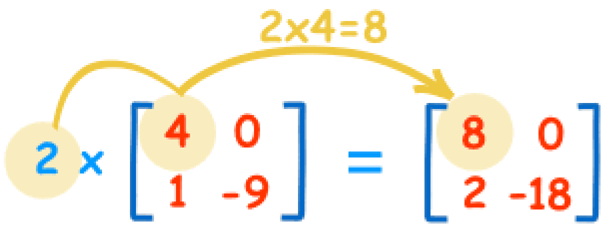
- For a $m\times k$ matrix **A** and a $k\times n$ matrix **B**, the *product* **AB** is the $m\times n$ matrix:

$$\mathbf{AB} = \mathbf{C} = [c_{i,j}] \equiv \left[\sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j} \right]$$

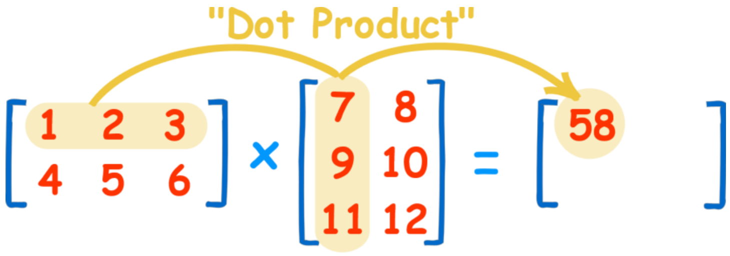
- *I.e.*, element (i,j) of **AB** is given by the vector *dot product* of the *i*th row of **A** and the *j*th column of **B** (considered as vectors).
- Note: Matrix multiplication is **not** commutative!

Matrix Product Example

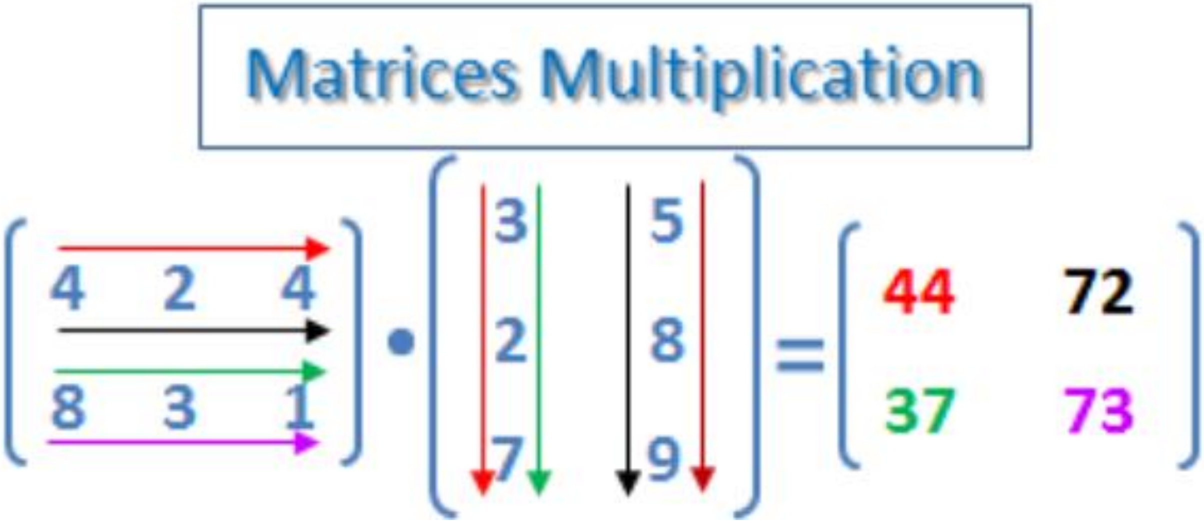
To multiply a matrix by a single number is easy:



But to multiply a matrix **by another matrix** we need to do the "dot product" of rows and columns ... what does that mean? Let us see with an example:
To work out the answer for the **1st row** and **1st column**:



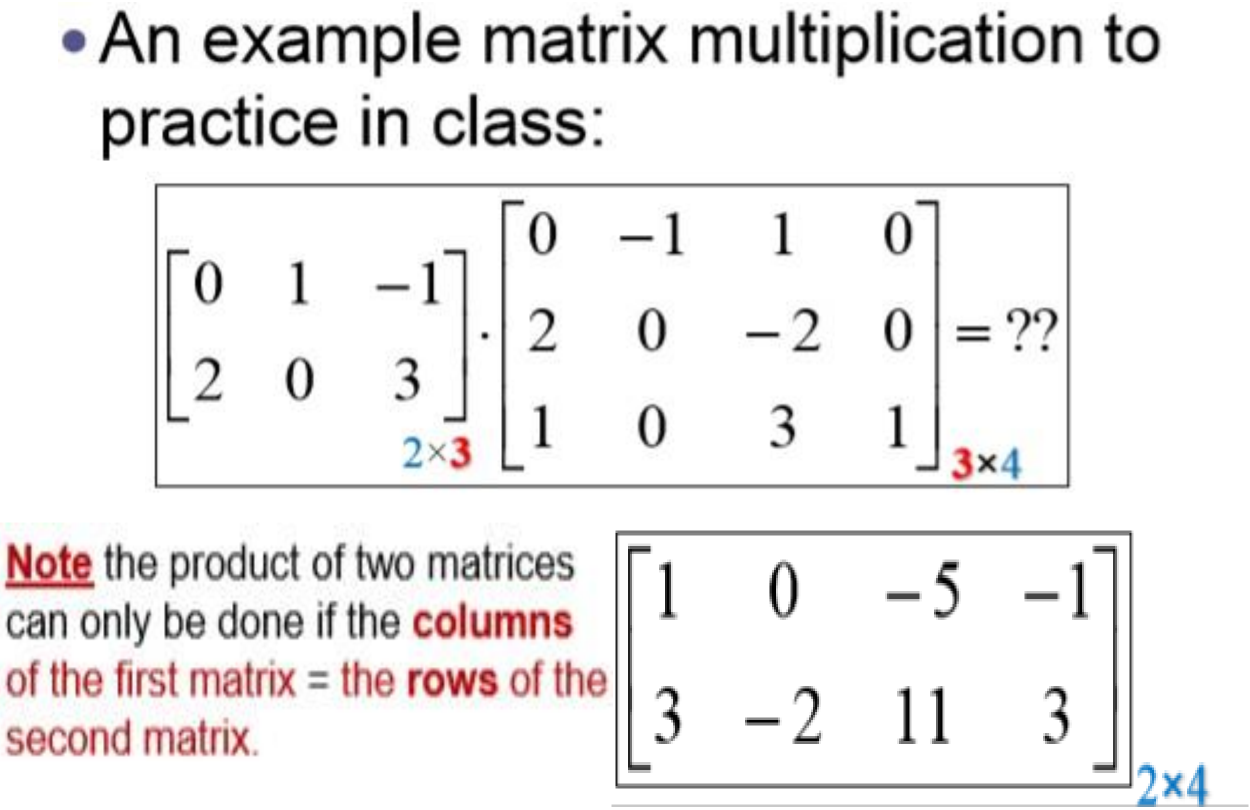
Matrix Product Example



Matrices Multiplication

$$\begin{pmatrix} 4 & 2 & 4 \\ 8 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ 2 & 8 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 44 & 72 \\ 37 & 73 \end{pmatrix}$$

Matrix Product Example



- An example matrix multiplication to practice in class:

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}_{2\times 3} \cdot \begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}_{3\times 4} = ??$$

Note the product of two matrices can only be done if the **columns** of the first matrix = the **rows** of the second matrix.

$$\begin{bmatrix} 1 & 0 & -5 & -1 \\ 3 & -2 & 11 & 3 \end{bmatrix}_{2\times 4}$$

Identity Matrices

- The ***identity matrix*** *of order* n, $\mathbf{I}_n$, is the order-n matrix with 1's along the **upper-left to lower-right diagonal** and 0's **everywhere else**. $A\,\mathbf{I}_n = A$

$$\mathbf{I}_n = \left[\begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Powers of Matrices

If **A** is an $n \times n$ square matrix and $p \geq 0$, then:

- $\mathbf{A}^p \equiv \underbrace{\mathbf{AAA} \cdots \mathbf{A}}_{p \text{ times}} \quad (\mathbf{A}^0 \equiv \mathbf{I}_n)$
- Example:

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^3 = \overbrace{\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}}^{p \text{ times}}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$$

Matrix Transposition

- If **A** is an *$m \times n$* matrix, the *transpose* of **A** is the *$n \times m$* matrix given by $\mathbf{A}^t$

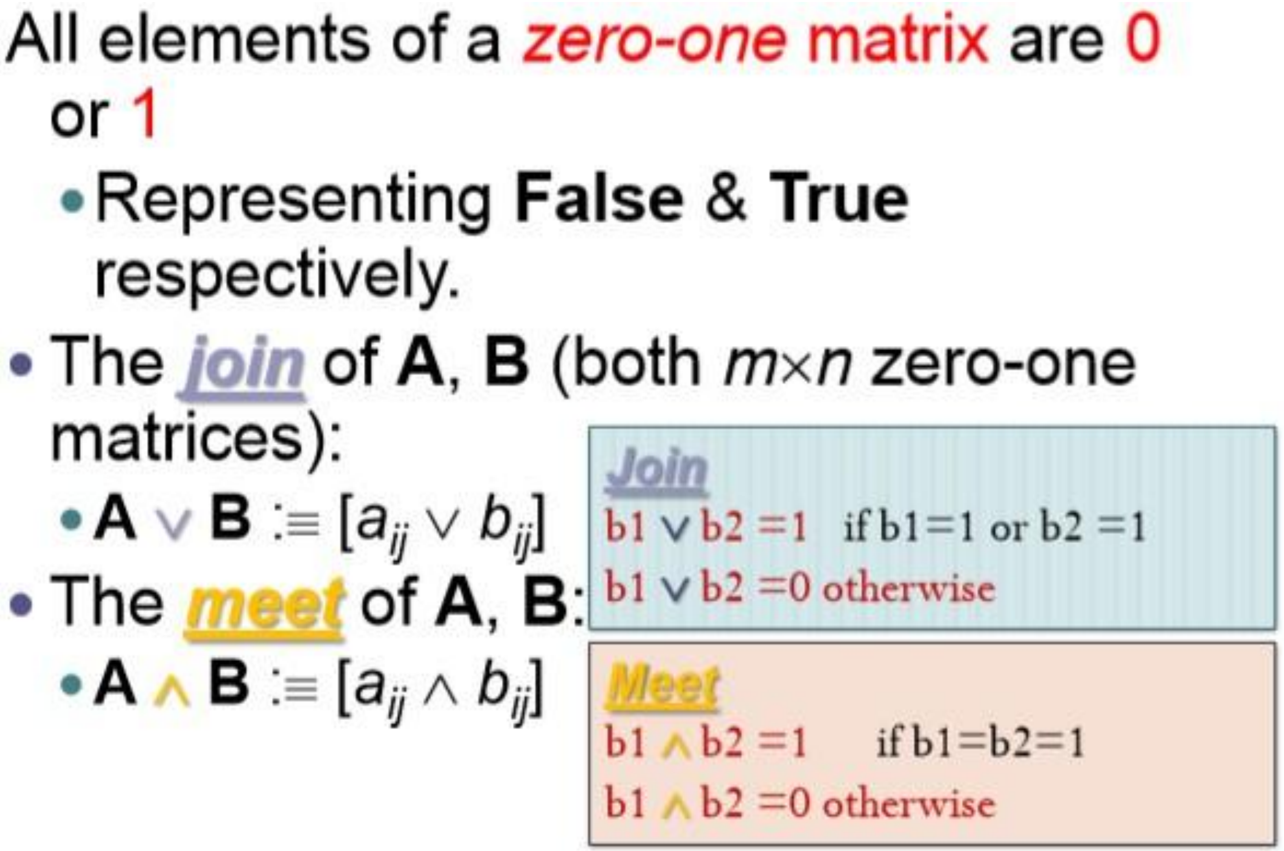
$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}^t = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}$$

Symmetric Matrices

- A square matrix **A** is *symmetric* iff **A**=**A**^{t}.
- Which is symmetric?

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \boxed{\begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & 2 \end{bmatrix}} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

Zero-One Matrices



All elements of a *zero-one matrix* are 0 or 1

- Representing **False** & **True** respectively.
- The *join* of **A**, **B** (both $m\times n$ zero-one matrices):
  - $\mathbf{A} \vee \mathbf{B} :\equiv [a_{ij} \vee b_{ij}]$
- The *meet* of **A**, **B**:
  - $\mathbf{A} \wedge \mathbf{B} :\equiv [a_{ij} \wedge b_{ij}]$

Join

$b1 \vee b2 = 1$ if $b1 = 1$ or $b2 = 1$

$b1 \vee b2 = 0$ otherwise

Meet

$b1 \wedge b2 = 1$ if $b1 = b2 = 1$

$b1 \wedge b2 = 0$ otherwise

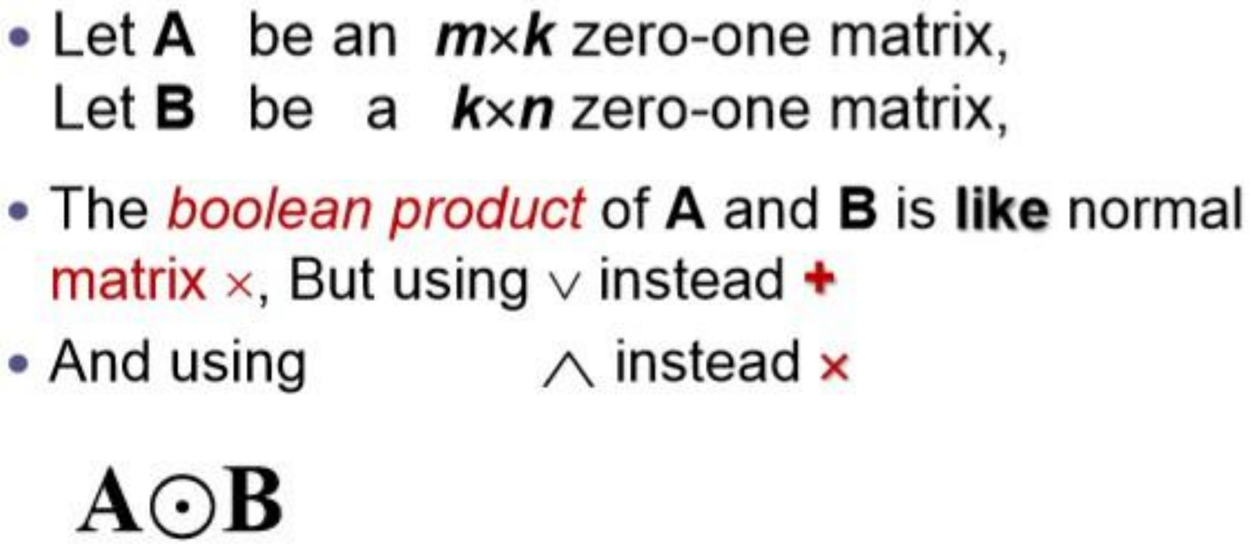
Example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

We find the join between $A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

We find the meet between $A \wedge B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Boolean Products



- Let **A** be an $m \times k$ zero-one matrix,
  Let **B** be a $k \times n$ zero-one matrix,
- The *boolean product* of **A** and **B** is **like** normal matrix $\times$, But using $\vee$ instead $+$
- And using $\wedge$ instead $\times$

$$\mathbf{A} \odot \mathbf{B}$$

Boolean Powers

- For a square zero-one matrix **A**, and any $k \geq 0$, the *k^{th} Boolean power of **A*** is simply the Boolean product of *k* copies of **A**.
- $\mathbf{A}^{[k]} \equiv \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{k \text{ times}}$
- (see page 254 Example 11)

Example

∨ instead +

∧ instead ×

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ B= $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$A \odot B = \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \end{bmatrix}$$

Then:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$