The Foundations: Logic and Proofs

CHAPTER 1

Chapter Summary

- Propositional Logic
  - The Language of Propositions
  - Applications
  - Logical Equivalences
- Predicate Logic
  - The Language of Quantifiers
  - Logical Equivalences
  - Multiple Quantifiers
- Proofs
  - Rules of Inference

Propositional Logic Summary

- The Language of Propositions
  - Connectives
  - Truth Values
  - Truth Tables
- Applications
  - Translating English Sentences
  - System Specifications
  - Logic Puzzles
  - Logic Circuits
- Logical Equivalences
  - Important Equivalences
  - Showing Equivalence
  - Satisfiability

Propositional Logic

SECTION 1.1

Section Summary

- Propositions
- Connectives
  - Negation
  - Conjunction
  - Disjunction
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth Tables

Logic

- Logic is the study of consequence.

  - Given a few mathematical statements or facts, we would like to be able to draw some conclusions.

- Logic aims to determine in which cases a conclusion is, or is not, a consequence of a set of premises.

- Example:
  - Premise/Fact #1: If Sara eats her vegetables, then she can have a cookie.
  - Premise/Fact #2: Sara eats her vegetables.
  - Conclusion: Sara gets a cookie.

C++ Example

```cpp
#include <iostream>
using namespace std;

main() {
   int a = 5;
   int b = 20;
   int c ;

   if(a && b) {
      cout << "Line 1 - Condition is true"<< endl ;
   }

   if(a || b) {
      cout << "Line 2 - Condition is true"<< endl ;
   }

   /* Let's change the values of  a and b */
   a = 0;
   b = 10;

   if(a && b) {
      cout << "Line 3 - Condition is true"<< endl ;
   } else {
      cout << "Line 4 - Condition is not true"<< endl ;
   }

   if(!(a && b)) {
      cout << "Line 5 - Condition is true"<< endl ;
   }

   return 0;
}
```

Output:

```
Line 1 - Condition is true
Line 2 - Condition is true
Line 4 - Condition is not true
Line 5 - Condition is true
```

Propositions

- A ***proposition*** is a declarative sentence that is either true or false.
- Examples of propositions:
  a) The Moon is made of green cheese.
  b) Cairo is the capital of Egypt.
  c) Toronto is the capital of Canada.
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

Propositional Logic

- The area of logic that deals with propositions is called the propositional calculus or **propositional logic**.
  - It was first developed systematically by the Greek **Philosopher Aristotle** more than 2300 years ago.

- Constructing Propositions
  - Propositional Variables: *p*, *q*, *r*, *s*, …
  - The proposition that is always true is denoted by **T** and
  - the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from **logical connectives** and other propositions
    - Conjunction ∧
    - Disjunction ∨
    - Negation ¬
    - Implication →
    - Biconditional ↔

Conjunction

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**: If p denotes **"I am at home."** and q denotes **"It is raining."** then $p \wedge q$ denotes **"I am at home and it is raining."**

Disjunction

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**:
- If p denotes **"I am at home."** and q denotes **"It is raining."** then $\boldsymbol{p \vee q}$ denotes **"I am at home or it is raining."**

The Connective Or in English

- In English "or" has two distinct meanings.
  - **"Inclusive Or"** - In the sentence **"Students who have taken CS202 or Math120 may take this class,"** we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \vee q$ to be true, either one or both of p and q must be true.
  - **"Exclusive Or"** - When reading the sentence **"Soup or salad comes with this entrée,"** we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	¬p
T	F
F	T

- **Example**: If p denotes **“The earth is round.”,** then $\neg p$ denotes **“It is not the case that the earth is round,”** or more simply **“The earth is not round.”**

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes **"I am at home."** and q denotes **"It is raining."** then $p \rightarrow q$ denotes **"If I am at home then it is raining."**
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Understanding Implication (cont)

- In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.

- These implications are perfectly fine, but would not be used in **ordinary English**.
  - "If the moon is made of green cheese, then I have more money than Bill Gates. "
  - "If the moon is made of green cheese then I'm on welfare."
  - "If 1 + 1 = 3, then your grandma wears combat boots."

Understanding Implication (cont)

- **Implication ‘If...then...’ Symbol: → Terminology**
- •**p**= premise, hypothesis, antecedent
- •**q** = conclusion, consequence

Different Ways of Expressing $p \rightarrow q$

if p**, then** q

if p, q

q **unless** $\neg p$

q **if** p

q **whenever** p

q **follows from** p

a necessary condition for p **is** q

a sufficient condition for q **is** p

p **implies** q

p **only if** q

q **when** p

p **is sufficient for** q

q **is necessary for** p

Converse, Contrapositive, and Inverse

- **From $p \rightarrow q$ we can form new conditional statements .**
- **Converse:** A proposition formed by interchange of the hypothesis and conclusion of a given proposition
  - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
- **Inverse:** A proposition formed by contradicting both the hypothesis and conclusion of a given proposition or theorem
  - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$
- **Contrapositive:** A proposition formed by contradicting both the hypothesis and conclusion of a given proposition and interchanging them
  - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$

Converse, Contrapositive, and Inverse

- **From $p \rightarrow q$ we can form new conditional statements .**
  - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
  - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$

  - **Example**: Find the converse, inverse, and contrapositive of "It is raining is a sufficient condition for me not going to town."

Solution:

p: It is raining q: not going to town

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$
- **read as "p if and only if q."** The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table: combining the symbols → and ⟵

p	q	p ⟷ q
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes **"I am at home."** and q denotes **"It is raining."** then $p \leftrightarrow q$ denotes **"I am at home if and only if it is raining."**
- p ↔q means that p&q have the same truth value.
- Note this truth table is the exact opposite of ⊕
- Thus, p ↔q means ¬(p ⊕ q)

Biconditional vs Exclusive Or

BICONDITIONAL - TRUTH TABLE

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	F	F
F	T	F
F	**F**	**T**

EXCLUSIVE OR - TRUTH TABLE

p	q	$p \oplus q$
T	**T**	**F**
T	F	T
F	T	T
F	**F**	**F**

Expressing the Biconditional

- Some alternative ways "*p* if and only if *q*" is expressed in English:

  - *p* **is necessary and sufficient for** *q*
  - **if** *p* **then** *q* **, and conversely**
  - *p* **iff** *q (abbreviation of* if and only if)

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for **every possible combination of values** for the atomic **propositions**.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the **truth value** of each expression that occurs in the compound proposition as it is built up.
    - This includes the atomic propositions

Example Truth Table

- Construct a truth table for $p \vee q \to \neg r$

Example Truth Table

- Construct a truth table for $p \vee q \to \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

- Construct the truth table of the compound proposition:

$$p \vee \neg q \rightarrow p \wedge q$$

Truth Tables of Compound Propositions

- Compound propositions involve any number of propositional variables and logical connectors.
- Construct the truth table of the **compound proposition**:

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

p	q	$\neg q$	$\neg p$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
F	F	T	T	T	F	F
F	T	F	T	F	F	T
T	F	T	F	T	F	F
T	T	F	F	T	T	T

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Equivalent Propositions

- Two propositions are ***e**quivalent* if they always have the same truth value.
- **Example**: Show using a truth table that the implication is equivalent to the contrapositive.

Solution:

p	q	¬ p	¬ q	p →q	¬q → ¬ p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the **converse** nor **inverse** of an **implication** are not equivalent to the implication.

Solution:

p	q	¬ p	¬ q	p →q	¬ p →¬ q	q → p
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

- How many rows are there in a truth table with *n* propositional variables?

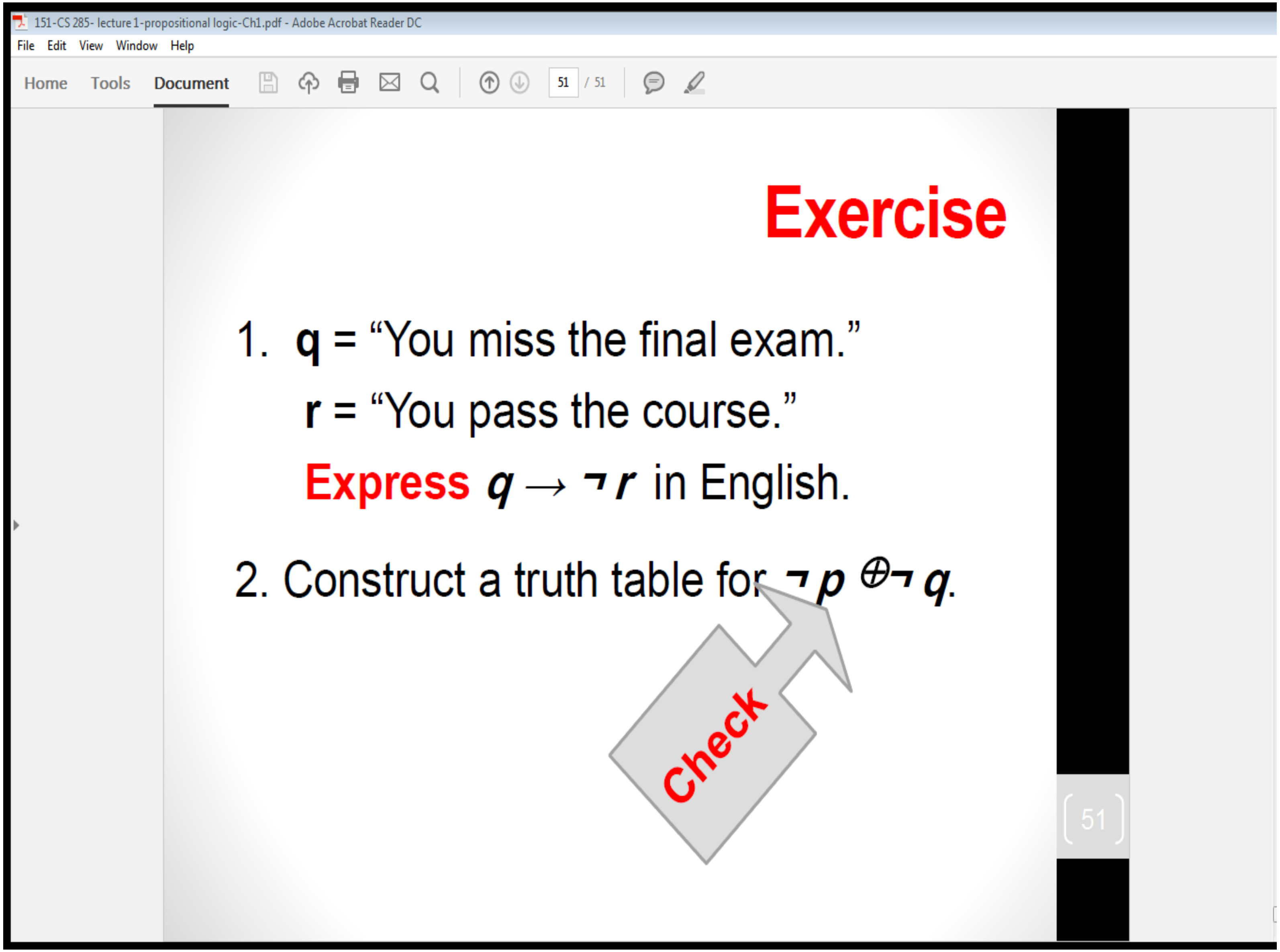
Solution: 2^n We will see how to do this in Chapter 6.

- Note that this means that with n propositional variables, we can construct 2^{2^n} distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
If the intended meaning is $p \vee (q \rightarrow \neg r)$
then parentheses must be used.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Exercise
1. q = “You miss the final exam.”
r = “You pass the course.”
Express q → ¬ r in English.
2. Construct a truth table for ¬ p ⊕ ¬ q.
Check
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Applications of Propositional Logic

SECTION 1.2

Applications of Propositional Logic: Summary

- Translating English to Propositional Logic
- System Specifications
- Logic Puzzles
- Logic Circuits

Translating English Sentences

- Statements in mathematics and the sciences and in natural language often are imprecise or ambiguous.
- To make such statements precise, they can be translated into the language of logic.
- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives
- "If I go to Harry's or to the country, I will not go shopping."
  - *p*: I go to Harry's
  - q: I go to the country.
  - *r*: I will go shopping.

If *p* or *q* then not *r*.

$$(p \lor q) \to \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

"**if** you are a computer science major **or** you are not a freshman, You can access the Internet from campus."

Solution:

Let *a*: "You can access the internet from campus,"

c: "You are a computer science major,"

f: "You are a freshman."

$$(c \lor \neg f) \rightarrow a$$

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent **when** the file system is full"

Solution: One possible solution:

Let q denote "The automated reply **can be** sent"

and p denote "The file system is full."

$$p \rightarrow \neg q$$

Consistent System Specifications

Definition: A list of propositions is ***consistent*** if it is possible to assign **truth values** to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution:

- Let **p** denote **"The diagnostic message is stored in the buffer."**
- Let **q** denote **"The diagnostic message is retransmitted"**
- The specification can be written as: $p \lor q, \neg p, p \to q$.
- When p is false and q is true all **three statements are true**. (see TT next)

So the specification is **consistent**.

Consistent System Specifications

p	q	$\neg p$	$p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T

Consistent System Specifications

What if "The diagnostic message is not retransmitted" is added?

Solution:

Now we are adding $\neg q$ and there is no satisfying assignment.

So the specification is **not consistent**.

p	q	¬p	p ∨q	p →q	¬q
T	T	F	T	T	F
T	F	F	T	F	T
F	T	T	T	T	F
F	F	T	F	T	T

Consistency and Inconsistency

- Consider the following set of propositions. Do they form a consistent or inconsistent set?

  (1) John is at the store.

  (2) Corinne plays guitar.

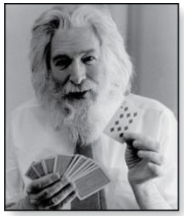
  (3) Mike cheats on his taxes.

  (4) John is a fire-fighter.

Solution:

- Consistent. Notice that nothing in the logical form of (1)-(4) prevents all of the propositions from being true.

Logic Puzzles



Raymond Smullyan (Born 1919)

- An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

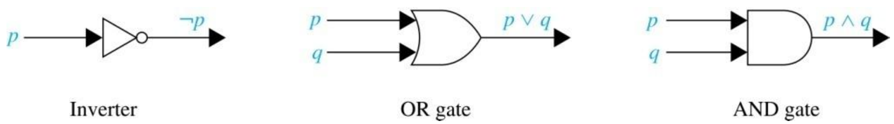
Example: What are the types of A and B?

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

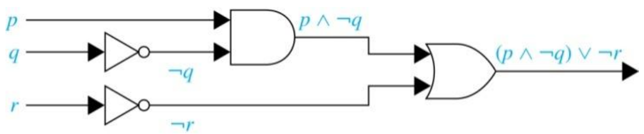
1. If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
2. If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Logic/digital Circuits

- Electronic circuits; each input/output signal can be viewed as a 0 or 1.
  - 0 represents **False**
  - 1 represents **True**
- Complicated circuits are constructed from **three basic circuits** called **gates**.



  - The inverter (**NOT gate**)takes an input bit and produces the negation of that bit.
  - The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
  - The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.
- More complicated digital circuits can be constructed by combining these 3 basic circuits to produce the desired output given the input signals by building a circuit for each piece of the output expression and then combining them. For example:



Propositional Equivalences

SECTION 1.3

Section Summary

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- Propositional Satisfiability
  - Sudoku Example

Tautologies, Contradictions, and Contingencies

- A **tautology** is a compound proposition which is always true.
  - Example: $p \lor \neg p$
- A ***contradiction*** is a proposition which is always false.
  - Example: $p \land \neg p$
- A ***contingency*** is a proposition which is neither a tautology nor a contradiction, such as p

P	¬p	p ∨¬p	p ∧¬p
T	F	T	F
F	T	T	F

Logically Equivalent

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- Two way to prove equivalence..

Logically Equivalent

- Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent

Logically Equivalent

- Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent
- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	¬p	¬p ∨ q	p→ q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	¬p	¬q	(p∨q)	¬(p∨q)	¬p∧¬q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T,$ $p \wedge F \equiv F$
- Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T,$ $p \wedge \neg p \equiv F$

Key Logical Equivalences (*cont*)

- Commutative Laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$
- Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive Laws: $(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$
  $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
- Absorption Laws: $p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$
\begin{aligned}
A &\equiv A_1 \\
&\vdots \\
A_n &\equiv B
\end{aligned}
$$

- Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law} \\
& && \text{for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$

is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and} \\
& && \text{commutative laws} \\
& && \text{laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

Equivalent Propositions

- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology using the truth table
- **Solution:**

p	q	$(p \wedge q)$	$(p \vee q)$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

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Tautology example – Part 1

Demonstrate that

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

is a **tautology** in two ways:

1. Using a truth table – show that

   $[\neg p \wedge (p \vee q)] \rightarrow q$ is always true

2. Using a proof (will get to this later).

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Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	q	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	F	T	F	F	F	T

Derivational Proof Techniques

- When a compound proposition involves many atomic components, the size of the truth table for the compound proposition becomes large
- Q1: How many rows are required to construct the truth-table of:

$((q \leftrightarrow (p \rightarrow r)) \land (\neg(s \land r) \lor \neg t)) \rightarrow (\neg q \rightarrow r)$?

- Q2: How many rows are required to construct the truth-table of a proposition involving n atomic components?
- A1: 32 rows, each additional variable doubles the number of rows
- A2: In general, 2^n rows
- Therefore, as compound propositions grow in complexity, truth tables become more and more unwieldy.
- In such cases, it is better to use derivational proof techniques

151-CS 285- lecture 2-propositional equivalences-Ch1.pdf - Adobe Acrobat Reader DC

Derivational Proof Techniques

EG: consider the compound proposition

$$(p \rightarrow p) \vee (\neg(s \wedge r) \vee \neg t)) \vee (\neg q \rightarrow r)$$

Q: Why is this a tautology?

Solution

Derivational Proof Techniques

A: Part of it is a tautology $(p \rightarrow p)$ and the disjunction of True with any other compound proposition is still True:

$$(p \rightarrow p) \lor (\neg(s \land r) \lor \neg t)) \lor (\neg q \rightarrow r)$$

$$\Leftrightarrow \quad \mathbf{T} \lor (\neg(s \land r) \lor \neg t)) \lor (\neg q \rightarrow r)$$

$$\Leftrightarrow \quad \mathbf{T}$$

Derivational techniques formalize the intuition of this example.

Propositional Satisfiability

- A compound proposition is ***satisfiable*** if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, the compound proposition is *unsatisfiable*.

- A compound proposition is unsatisfiable if and only if its **negation is a tautology.**

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Solution: Satisfiable. Assign **T** to p, q, and r.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to p and $\boldsymbol{F}$ to q.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

Predicates and Quantifiers

SECTION 1.4

Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan’s Laws for Quantifiers
- Translating English to Logic

Propositional Logic Not Enough

- If we have:

  "All men are mortal."

  "Socrates is a man."

- Does it follow that "Socrates is mortal?"
- Can't be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.

Introducing Predicate Logic

- The statement **"*x* is greater than 3"** has two parts.
  - The first part, the **variable** *x*, is the subject of the statement.
  - The second part—the **predicate**, "is greater than 3"—refers to a property that the subject of the statement can have.
- We can denote the statement "*x* is greater than 3" by ***P(x)***, where *P* denotes the predicate "is greater than 3" and *x* is the variable.
- The statement *P(x)* is also said to be the value of the **propositional function** *P* at *x*.
- Once a value has been assigned to the variable *x*, the statement *P(x)* becomes a **proposition** and has a truth value:
  - **What is the truth values of *P(4)?*** by setting $x = 4$ in the statement "$x > 3$." Hence, *P*(4), which is the statement "$4 > 3$," is **true**.
  - **What is the truth values of *P(2)*?** *P*(2), which is the statement "$2 > 3$," is **false**.

Introducing Predicate Logic

- A generalization of propositions **- *propositional functions*** or ***predicates***: propositions which contain variables

- **Predicates** become **propositions** once every variable is bound- by:
  - Assigning it a value from the *Universe of Discourse* U

    or
  - Quantifying it

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: e.g. x, y, z
  - Predicates: e.g. equals 10, is green
  - Propositional functions : e.g $P(x)$, $M(x)$
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They combine variables and a predicate, e.g., $P(x)$ = x is mortal
  - Variables can be replaced by elements from their *domain*.

Propositional Functions

- **Propositional functions** become **propositions** (and have truth values) when their variables are replaced by a value from the ***domain*** (or *bound* by a quantifier, as we will see later).
- The statement *P(x)* is said to be the value of the propositional function *P* at *x*.
- For example, let *P(x)* denote "$x > 0$" and the domain be the integers. Then:

  P(-3) is false.

  P(0) is false.

  P(3) is true.

- Often the domain is denoted by *U*. So in this example *U* is the integers.

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  Solution: F

  R(3,4,7)

  Solution: T

  R(x, 3, z)

  Solution: Not a Proposition

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  Q(2,-1,3)

  Solution: T

  Q(3,4,7)

  Solution: F

  Q(x, 3, z)

  Solution: Not a Proposition

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If *P(x)* denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$

- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier,* **"**For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for every x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for some x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier (∀)

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Existential Quantifier(∃)

- $\exists x\, P(x)$ is read as "For some x, $P(x)$", or as "There is an x such that $P(x)$," or "For at least one x, $P(x)$."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Uniqueness Quantifier (*optional*)

- **$\exists! x\, P(x)$** means that $P(x)$ is true for <u>one and only one</u> x in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$ is true."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\, P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\, P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique x such that $P(x)$ can be expressed as:

$$\exists x\, (P(x) \land \forall y\, (P(y) \rightarrow y = x))$$

Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the **loops will not terminate in some cases**.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- **Examples**:
    1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
    2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
    3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
        - But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \lor Q(x)$ means $(\forall x\, P(x)) \lor Q(x)$
- $\forall x\, (P(x) \lor Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \lor Q(x)$ when they mean $\forall x\, (P(x) \lor Q(x))$.

Predicates & Quantifiers

- **REMEMBER!**
  - A predicate is <u>not</u> a proposition until <u>*all*</u> variables have been bound either by quantification or assignment of a value!

Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain as all people.
- The two premises are: $$\forall x Man(x) \rightarrow Mortal(x)$$ $$Man(Socrates)$$
- The conclusion is: $$Mortal(Socrates)$$
- Later we will show how to prove that the conclusion follows from the premises.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic:

"Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class,

- define a propositional function J(x) denoting "x has taken a course in Java" and
- translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also

- define a propositional function S(x) denoting "x is a student in this class" and
- translate as $\forall x\, (S(x) \rightarrow J(x))$.

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic:

"Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *U* is all people, then translate as

$$\exists x\, (S(x) \land J(x))$$

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for **every predicate** substituted into these statements and
  - for **every domain** of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example**: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a **universally quantified proposition** is equivalent to a **conjunction of propositions** without quantifiers and an **existentially quantified proposition** is equivalent to a **disjunction of propositions** without quantifiers.
- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is the statement "x has taken a course in Java" and

  the **domain** is students in your class.

- Negating the original statement gives **"It is not the case that every student in your class has taken Java."** This is equivalent to **"There is a student in your class who has not taken Java."**

  Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives **"It is not the case that there is a student in this class who has taken Java."** This is equivalent to **"Every student in this class has not taken Java"**

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

Multiple Quantifiers

- Multiple Quantifiers: read left to right . . .
- Example1. Let U = R, the real numbers, P(x,y): xy= 0

$$\forall x\, \forall y\, P(x, y)$$
$$\forall x\, \exists y\, P(x, y)$$
$$\exists x\, \forall y\, P(x, y)$$
$$\exists x\, \exists y\, P(x, y)$$

  - The only one that is false is the first one.

Multiple Quantifiers

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Multiple Quantifiers

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x \ (S(x) \wedge M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x \ (S(x) \rightarrow (M(x) \vee C(x)))$$

Some Fun with Translating from English into Logical Expressions

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): x is a thingamabob

  Translate "Everything is a fleegle"

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

  “Nothing is a snurd.”

Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x \neg S(x)$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

"All fleegles are snurds."

Solution: $\forall x\,(F(x) \rightarrow S(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\ (F(x) \land T(x))$

Translation (cont)

- U = {fleegles, snurds, thingamabobs }

  $F(x)$: x is a fleegle

  $S(x)$: x is a snurd

  $T(x)$: x is a thingamabob

  "No snurd is a thingamabob."

  Solution: $\neg\exists x\,(S(x) \land T(x))$

What is this equivalent to?

  Solution: $\forall x\,(\neg S(x) \lor \neg T(x))$

Translation

Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

- **"All hummingbirds are richly colored."**
- **"No large birds live on honey."**
- **"Birds that do not live on honey are dull in color."**
- **"Hummingbirds are small."**

Let:

- ***P(x)*, be the statements "*x* is a hummingbird,"**
- ***Q(x)*, be the statements "*x* is large,"**
- ***R(x)*, be the statements "*x* lives on honey,"**
- ***S(x)* be the statements "*x* is richly colored,"**

Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and *P(x)*, *Q(x)*, *R(x)*, and *S(x)*.

Solution:

We can express the statements in the argument as

- $\forall x(P(x) \rightarrow S(x))$.
- $\neg \exists x(Q(x) \wedge R(x))$.
- $\forall x(\neg R(x) \rightarrow \neg S(x))$.
- $\forall x(P(x) \rightarrow \neg Q(x))$.

Example

Use the following predicates to translate the sentences into predicate logic?

I(x): x has an internet
C(x,y): x and y have chatted
S(x): x is student

1. Ali and Ahmed has chatted
2. Ali is student and has internet connection
3. Sami or Ahmed has internet connection
4. If Sami is a student, then he has an internet connection
5. Someone has internet connection
6. Someone has chatted with everyone
7. Everyone has chatted with someone
8. No one has internet connection
9. A student has internet connection
10. Every student have internet connection
11. A student has chatted with Sami or with Ahmed

Rules of Inference

ROSEN 1.6

Rules of Inference

Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}$$

Proofs in mathematics are *valid arguments-* establish the truth of statements

An *argument* is a sequence of statements that end in a conclusion

By *valid* we mean the conclusion must follow from the truth of the preceding statements or premises

We use *rules of inference* to construct valid arguments

Valid Arguments in Propositional Logic

- Let p to represent = you have my password, q= you can log on to my email
- Fact 1:
  - "If you have my password, then you can log on to my email",
  - $p \to q = T$
- Fact 2:
  - "You cannot log on to my email",
  - $\neg q = T$
- What can we conclude from these two facts??
- **$\neg p = T$** You do not have a password!!!!

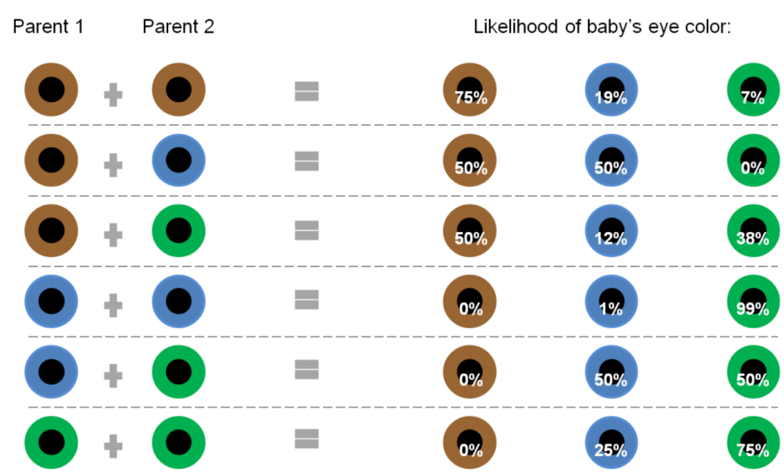
Valid Arguments in real life!

- Example 1:
  - Premise/Fact 1:
    - your blood type is O
  - Premise/Fact 2:
    - your partner blood type is O
  - Conclusion:
    - your children blood type must be O

- Example 2:
  - Premise/Fact 1:
    - your eye color is brown
  - Premise/Fact 2:
    - your partner eye color is blue
  - Conclusion:
    - your children can not have Green eyes.

Child's Blood Type

Mother's Blood Type \ Father's Blood Type	A	B	AB	O
A	A or O	A, B, AB, or O	A, B, or AB	A or O
B	A, B, AB, or O	B or O	A, B, or AB	B or O
AB	A, B, or AB	A, B, or AB	A, B, or AB	A or B
O	A or O	B or O	A or B	O



Valid Arguments in Propositional Logic

- Is this a valid argument?
  - If you listen you will hear what I'm saying
  - You are listening
  - Therefore, you hear what I am saying

Let **p** : you listen

Let **q** : you hear what I am saying

The argument has the form:

$$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

The Symbol $\therefore$
<u>Means</u>
Therefore

Valid Arguments in Propositional Logic

Show that $\neg(p \vee (\neg p \wedge q))$
is logically equivalent to $\neg p \wedge \neg q$

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

When we replace statements/propositions with propositional variables we have an ***argument form***.

Defn:
An argument (in propositional logic) is a sequence of propositions.
All but the final proposition are called ***premises***.
The last proposition is the ***conclusion***
The argument is valid iff the truth of all premises implies the conclusion is **true**
An argument form is a sequence of compound propositions

Valid Arguments in Propositional Logic

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

=

$$((p \rightarrow q) \wedge p) \rightarrow q$$

is a tautology (always true)

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Valid Arguments in Propositional Logic

The argument form with premises $p_1, p_2, \Lambda\ , p_n$

and conclusion q

is valid when $(p_1 \wedge p_2 \wedge \Lambda\ \wedge p_n) \rightarrow q$ is a tautology

We prove that an argument form is valid by using the laws of inference

But we could use a truth table. Why not?

Rules of Inference for Propositional Logic

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

modus ponens
aka
law of detachment

modus ponens (Latin) translates to "***mode that affirms***" modus ponens

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Rules of Inference for Propositional Logic

If it's a nice day we'll go to the beach. Assume the hypothesis "it's a nice day" is true. Then by modus ponens it follows that "we'll go to the beach".

Rules of Inference for Propositional Logic

$$\sqrt{2} > \frac{3}{2} \rightarrow (\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$$

$$\sqrt{2} > \frac{3}{2}$$

$$\therefore 2 > \left(\frac{3}{2}\right)^2$$

- **A valid argument can lead to an incorrect conclusion, if one of its premises is wrong/false!**

$$p:\ \sqrt{2} > \frac{3}{2}$$

$$q:\ \ 2 > \left(\frac{3}{2}\right)^2$$

$$p \rightarrow q$$

The argument is valid as it is constructed using modus ponens.
But one of the premises is false (q is false)
So, we cannot derive the conclusion is true. Also note that the conclusion q of this statement is false

The rules of inference

Rule of inference	Tautology	Name
$p \to q$ $\underline{p \qquad}$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $\underline{p \to q}$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollen
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p \quad}$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
$\underline{p \qquad}$ $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p $\underline{q \qquad}$ $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (p \vee r)$	Resolution

Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

Example:
Let *p* be "it is snowing."
Let *q* be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Corresponding Tautology:
$((p \to q) \land (q \to r)) \to (p \to r)$

Example:
Let *p* be "it snows."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$p \vee q$$
$$\neg p$$
$$\therefore q$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Example:

Let p be "I will study discrete math."
Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:

$(p \wedge q) \rightarrow p$

Example:

Let p be "I will study discrete math."

Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$

Corresponding Tautology:
$((p) \land (q)) \rightarrow (p \land q)$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:

$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:

Let *p* be "I will study discrete math."
Let *r* be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Using the rules of inference to build arguments

Show that the premises:

It is not sunny this afternoon and it is colder than yesterday.
If we go swimming it is sunny this afternoon.
If we do not go swimming then we will take a canoe trip.
If we take a canoe trip then we will be home by sunset.

lead to the conclusion:

We will be home by sunset

An example

Using the rules of inference to build arguments

1. It is not sunny this afternoon and it is colder than yesterday.
2. If we go swimming it is sunny this afternoon .
3. If we do not go swimming then we will take a canoe trip.
4. If we take a canoe trip then we will be home by sunset.
5. **We will be home by sunset**

p	It is sunny this afternoon
q	It is colder than yesterday
r	We go swimming
s	We will take a canoe trip
t	We will be home by sunset (the conclusion)

1. $\neg p \wedge q$
2. $r \rightarrow p$
3. $\neg r \rightarrow s$
4. $s \rightarrow t$
5. t

propositions

hypotheses

Using the rules of inference to build arguments

An example

p	It is sunny this afternoon
q	It is colder than yesterday
r	We go swimming
s	We will take a canoe trip
t	We will be home by sunset (the conclusion)

1. $\neg p \wedge q$
2. $r \rightarrow p$
3. $\neg r \rightarrow s$
4. $s \rightarrow t$
5. t

1.	¬p ∧ q	**Given**
2.	r → p	**Given**
3.	¬r→s	**Given**
4.	s →t	**Given**
5.	¬p	**Simplification(1)**
6.	¬r	**Modus Tollen(2,5)**
7.	s	**Modus Ponens(3,6)**
8.	t	**Modus Ponens(4,7)**

Rule of inference	Tautology	Name
$p \rightarrow q$ p $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollen
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (p \vee r)$	Resolution

Using the resolution rule (an example)

1. Anna is skiing or it is not snowing.
2. It is snowing or Bart is playing hockey.
3. Consequently Anna is skiing or Bart is playing hockey.

We want to show that (3) follows from (1) and (2)

Using the resolution rule (an example)

1. Anna is skiing or it is not snowing.
2. It is snowing or Bart is playing hockey.
3. Consequently Anna is skiing or Bart is playing hockey.

hypotheses

1. $p \vee \neg r$
2. $r \vee q$

propositions

p	Anna is skiing
q	Bart is playing hockey
r	it is snowing

1. $p \vee \neg r$
2. $r \vee q$

$$\frac{p \vee \neg r \qquad r \vee q}{p \vee q}$$

← Resolution rule

Consequently Anna is skiing or Bart is playing hockey

Example

- Here's what you know:
  1. Ellen is a math major or a CS major.
  2. If Ellen does not like discrete math, she is not a CS major.
  3. If Ellen likes discrete math, she is smart.
  4. Ellen is not a math major.
- Can you conclude Ellen is smart? **S**

Propositions

M Ellen is a math major
C Ellen is a CS major
D Ellen likes Discrete Math
S Ellen is smart

Hypotheses

1. $M \lor C$
2. $\neg D \rightarrow \neg C$
3. $D \rightarrow S$
4. $\neg M$

Continue

1.	$M \vee C$	Given
2.	$\neg D \rightarrow \neg C$	Given
3.	$D \rightarrow S$	Given
4.	$\neg M$	Given
5.	C	DS(1,4)
6.	D	MT(2,5)
7.	S	MP(3,6)

Rule of inference	Tautology	Name
$p \rightarrow q$, p, $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\neg q$, $p \rightarrow q$, $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollen
$p \rightarrow q$, $q \rightarrow r$, $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$, $\neg p$, $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
p, $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$, $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p, q, $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$, $\neg p \vee r$, $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (p \vee r)$	Resolution

Another view on what we are doing

You might think of this as some sort of game.

You are given some statement, and you want to see if it is a valid argument and true

You translate the statement into argument form using propositional variables, and make sure you have the premises right, and clear what is the conclusion

You then want to get from premises/hypotheses (A) to the conclusion (B) using the rules of inference.

So, get from A to B using as "moves" the rules of inference

Rule of inference	Tautology	Name
$p \to q$; p; $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$; $p \to q$; $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollen
$p \to q$; $q \to r$; $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$; $\neg p$; $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p; $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$; $\therefore p$	$(p \wedge q) \to p$	Simplification
p; q; $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$; $\neg p \vee r$; $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (p \vee r)$	Resolution

Example

- Show that the premises:
  1. $p \to q$
  2. $\neg p \to r$
  3. $r \to s$
- lead to the conclusion: $\neg q \to s$
- Hint: use logical equivalences and rules of inference.
  4. $\neg q \to \neg p$ Contrapositive of (1)
  5. $\neg q \to r$ Hypothetical syllogism using (4) and (2)
  6. $\neg q \to s$ Hypothetical syllogism using (5) and (3)

$$(p \vee (q \wedge r)) \equiv (p \vee q)) \wedge (p \vee r)$$

$$p \rightarrow q \equiv \neg p \vee q$$

$\frac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification

$p \vee q$ $\frac{\neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (p \vee r)$	Resolution

Example

- Show that the premises:
  1. $(\mathbf{p} \wedge \boldsymbol{q}) \vee \boldsymbol{r}$
  2. $\mathbf{r} \rightarrow \mathbf{s}$
- lead to the conclusion: $\boldsymbol{p} \vee \boldsymbol{s}$
- Hint: use logical equivalences and rules of inference.
  3. $(\mathrm{p} \vee r) \wedge (q \vee r)$ — Distributive law of (1)
  4. $\mathrm{p} \vee r$ — Simplification of (3)
  5. $\mathrm{q} \vee r$ — Simplification of (3)
  6. $\neg \mathrm{r} \vee s$ — Equivalent clause to (2)
  7. $\mathrm{p} \vee s$ — Resolution (4) and (6)