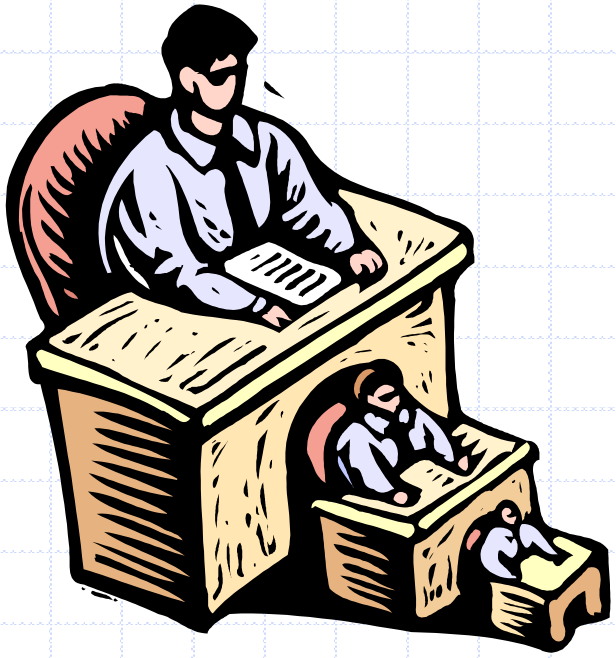
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

Recursion



When we stop loop ?

method call it self.

The Recursion Pattern

- Recursion: when a method calls itself
- Classic example – the factorial function:

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

$4! = 4*3*2*1$

$4! = 4*f(3)$

$3! = 3*f(2)$

$2! = 2*f(1)$

- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

base case

$n \geq 0$

- As a Java method:

```
1  public static int factorial(int n) throws IllegalArgumentException {
2    if (n < 0)                                   // base case
3      throw new IllegalArgumentException();      // argument must be nonnegative
4    else if (n == 0)
5      return 1;                                  // base case
6    else
7      return n * factorial(n-1);                 // recursive case
8  }
```

recursive call.

must update (n).

جملة الطباعة قبل الـ call بتكون forward.

جملة الطباعة بعد الـ call بتكون backward.

Content of a Recursive Method

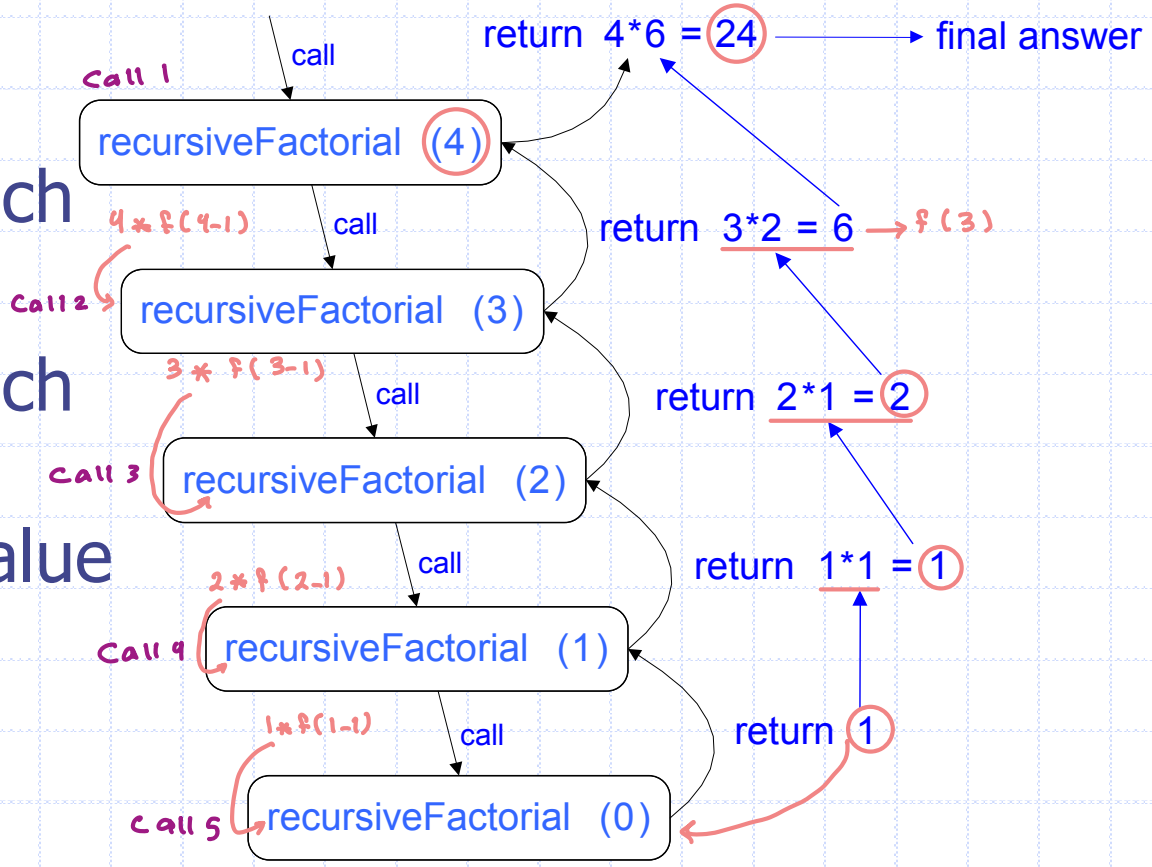
- Base case(s) ← must stop
  - Values of the input variables for which we perform no recursive calls are called base cases (there should be at least one base case).
  - Every possible chain of recursive calls must eventually reach a base case.
- Recursive calls method must call
  - Calls to the current method.
  - Each recursive call should be defined so that it makes progress towards a base case.

Visualizing Recursion

run time:
n=9
→ (9+1) calls
→ O(n)

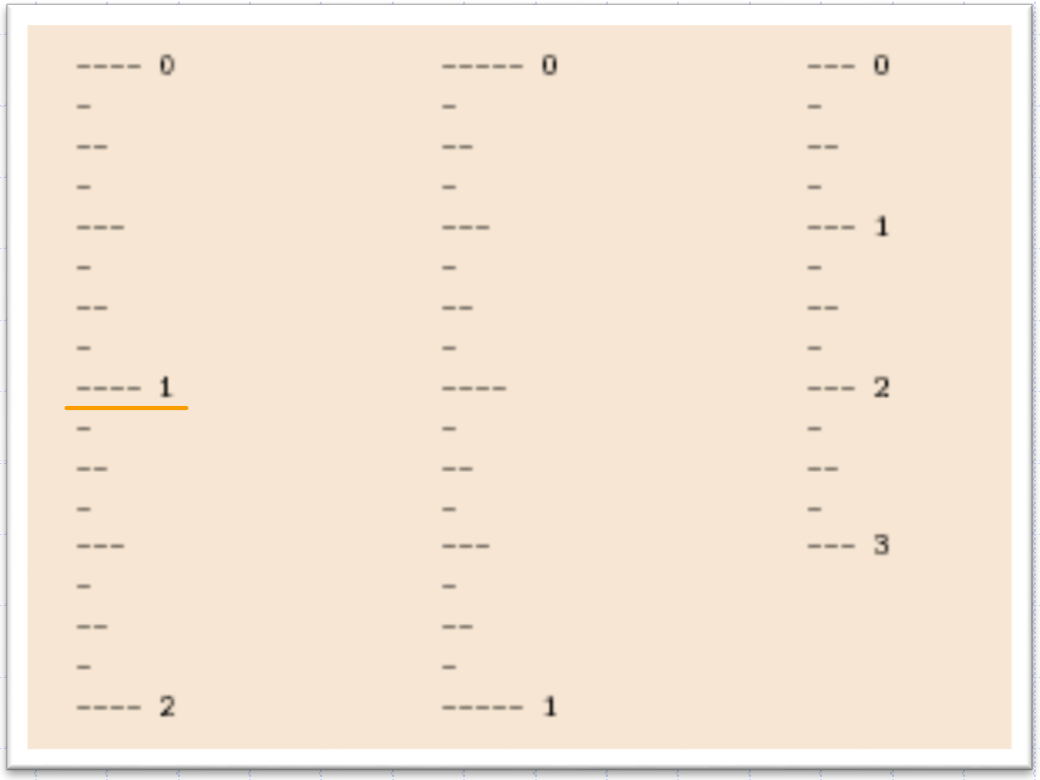
for (i=0 ; i<n ,i++)

- Recursion trace
  - A box for each recursive call
  - An arrow from each caller to callee
  - An arrow from each callee to caller showing return value
- Example



Example: English Ruler

- Print the ticks and numbers like an English ruler:



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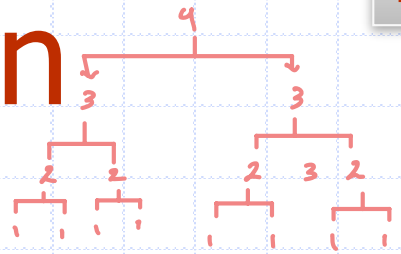
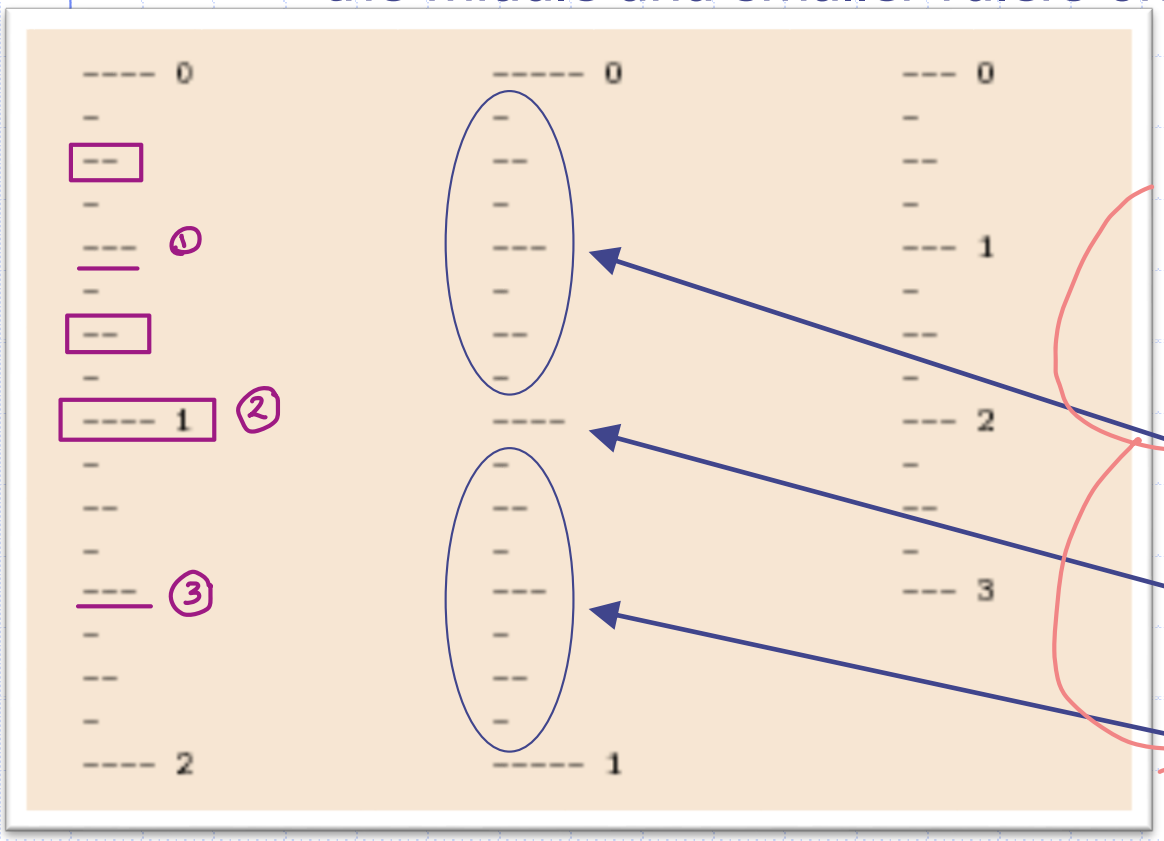
Using Recursion



drawInterval(length)

Input: length of a 'tick'

Output: ruler with tick of the given length in the middle and smaller rulers on either side



4 # ticks

drawInterval(length)

else →

if(length > 0) then ③

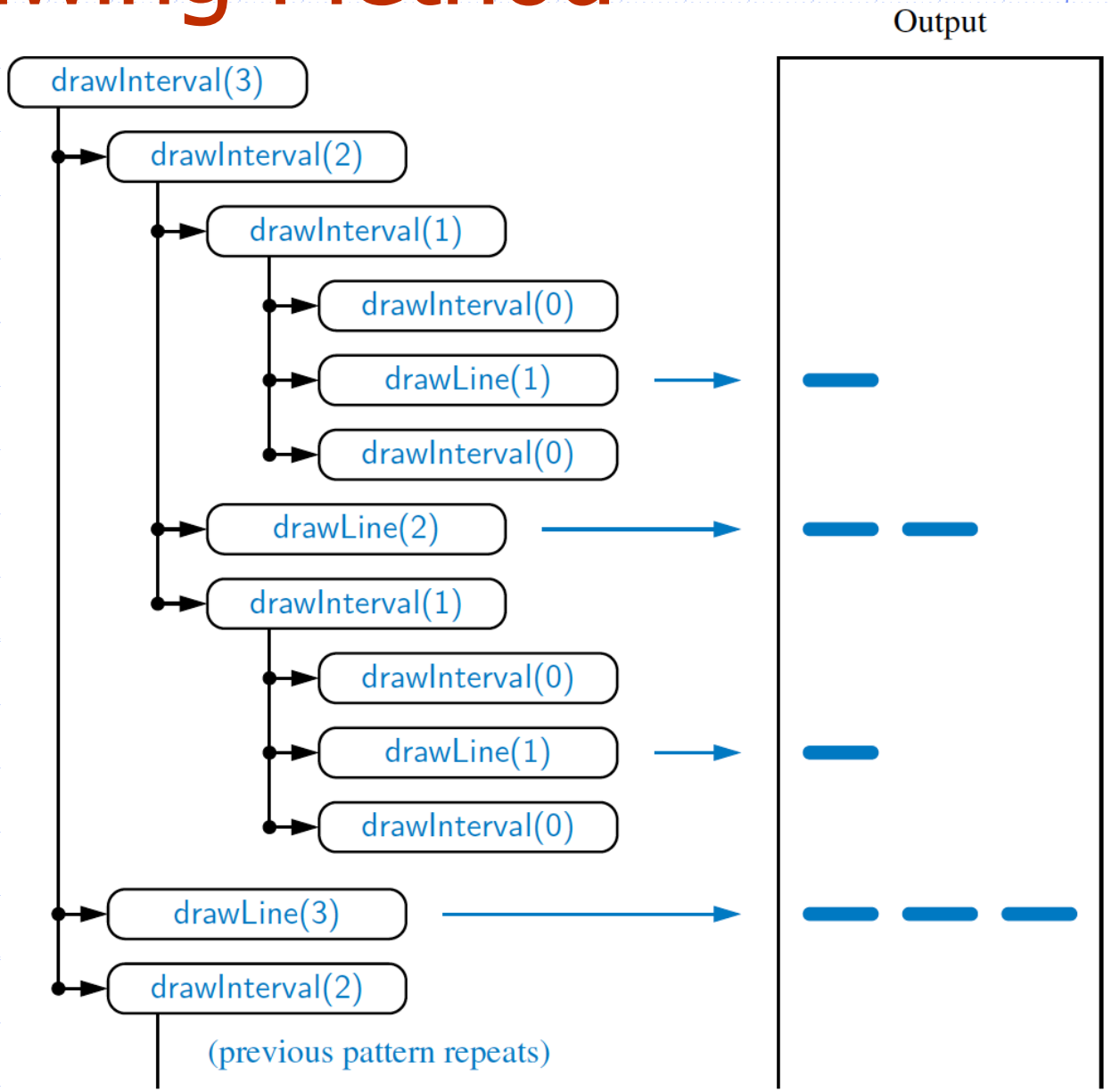
drawInterval (length – 1)

draw line of the given length

drawInterval (length – 1)

Recursive Drawing Method

- The drawing method is based on the following recursive definition
- An interval with a central tick length $L \geq 1$ consists of:
  - An interval with a central tick length $L-1$
  - An single tick of length L
  - An interval with a central tick length $L-1$



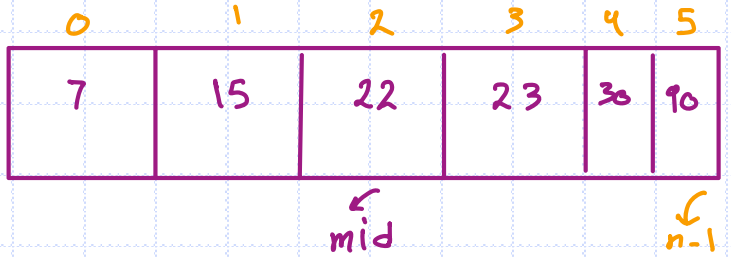
A Recursive Method for Drawing Ticks on an English Ruler

```
1  /** Draws an English ruler for the given number of inches and major tick length. */
2  public static void drawRuler(int nInches, int majorLength) {
3    drawLine(majorLength, 0);                 // draw inch 0 line and label
4    for (int j = 1; j <= nInches; j++) {
5      drawInterval(majorLength − 1);          // draw interior ticks for inch
6      drawLine(majorLength, j);               // draw inch j line and label
7    }
8  }
9  private static void drawInterval(int centralLength) {
10   if (centralLength >= 1) {                 // otherwise, do nothing
11     drawInterval(centralLength − 1);        // recursively draw top interval
12     drawLine(centralLength);                // draw center tick line (without label)
13     drawInterval(centralLength − 1);        // recursively draw bottom interval
14   }
15 }
16 private static void drawLine(int tickLength, int tickLabel) {
17   for (int j = 0; j < tickLength; j++)
18     System.out.print("-");
19   if (tickLabel >= 0)
20     System.out.print(" " + tickLabel);
21   System.out.print("\n");
22 }
23 /** Draws a line with the given tick length (but no label). */
24 private static void drawLine(int tickLength) {
25   drawLine(tickLength, −1);
26 }
```

Note the two recursive calls

binary recursive

search (17)



Binary Search

Search for an integer in an ordered list

= (low + high) / 2

17 ?? 22 → less than

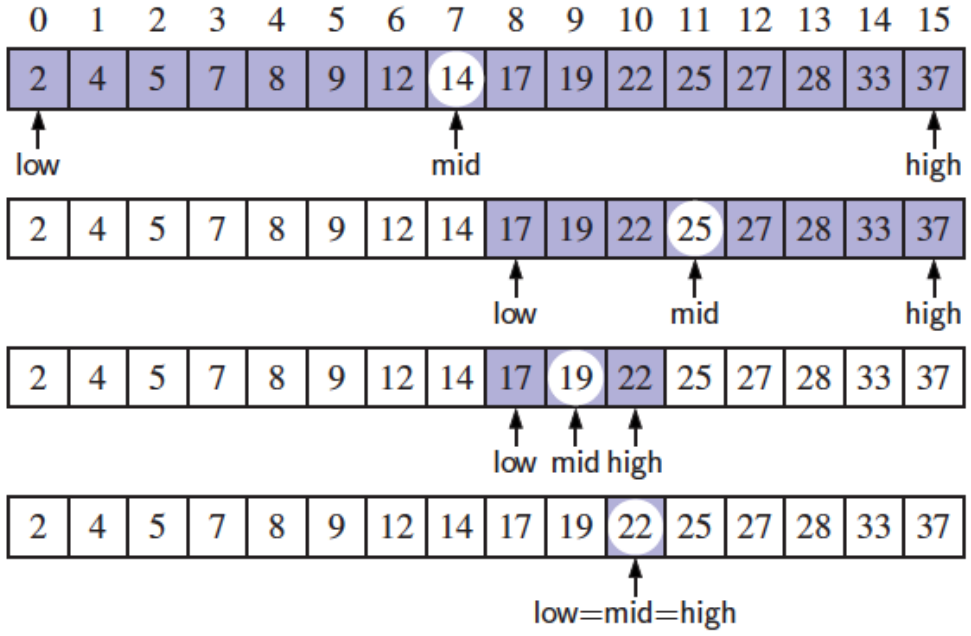
```
1  /**
2   * Returns true if the target value is found in the indicated portion of the data array.
3   * This search only considers the array portion from data[low] to data[high] inclusive.
4   */
5  public static boolean binarySearch(int[ ] data, int target, int low, int high) {
6    if (low > high)
7      return false;                          // interval empty; no match
8    else {
9      int mid = (low + high) / 2;
10     if (target == data[mid])
11       return true;                         // found a match
12     else if (target < data[mid])
13       return binarySearch(data, target, low, mid - 1);   // recur left of the middle
14     else
15       return binarySearch(data, target, mid + 1, high);  // recur right of the middle
16   }
17 }
```

mid = (0+1)/2 = 0, 1 ↳ choose one index.

find run time: $O(\log_2 n)$

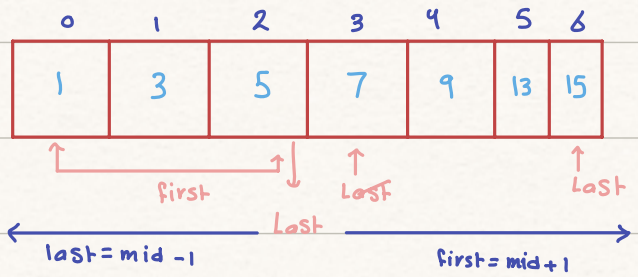
Visualizing Binary Search

- We consider three cases:
  - If the target equals data[mid], then we have found the target.
  - If target < data[mid], then we recur on the first half of the sequence.
  - If target > data[mid], then we recur on the second half of the sequence.



Binary Search

لازم الـ list تكون مرتبة.



مشينا ٣ متغيرات :

first = 0

last = 6

Mid = (first + last)/2 = (0+6)/2 = 3

7 ? = 6

first = 0

last = mid - 1 = 3-1 = 2

Mid = 0+2/2 = 1

3 ? = 6
<

first = mid + 1 = 2

last = 2

mid = 2+2/2 = 2

5 ? = 6
<

first = 2+1 = 3

last = 2

not found

Analyzing Binary Search

- Runs in O(log n) time.
  - The remaining portion of the list is of size high – low + 1
  - After one comparison, this becomes one of the following:

$$(\text{mid}-1)-\text{low}+1=\left\lfloor\frac{\text{low}+\text{high}}{2}\right\rfloor-\text{low}\leq\frac{\text{high}-\text{low}+1}{2}$$

$$\text{high}-(\text{mid}+1)+1=\text{high}-\left\lfloor\frac{\text{low}+\text{high}}{2}\right\rfloor\leq\frac{\text{high}-\text{low}+1}{2}.$$

  - Thus, each recursive call divides the search region in half; hence, there can be at most log n levels

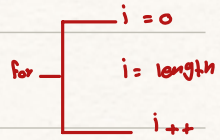
Linear Recursion

- Test for base cases
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls must eventually reach a base case, and the handling of each base case should not use recursion.
- Recur once
  - Perform a single recursive call
  - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
  - Define each possible recursive call so that it makes progress towards a base case.

دالة تطبع Array

3	4	7	9	17	19	15
0	1	2	3	4	5	6

لازم اعرف البداية والنهاية .




```
void
linear Print ( int data , first , int last ; i )
                 0        6          0
base {                              1
                                    2
   if ( i > last)                   6
      return ;                      7

   else
   {
                    3  4  7        15
      System.out.Println ( data [i] );

      linear Print ( data , first , last , i+1 );
```

كيف لو بطبع بالعكس

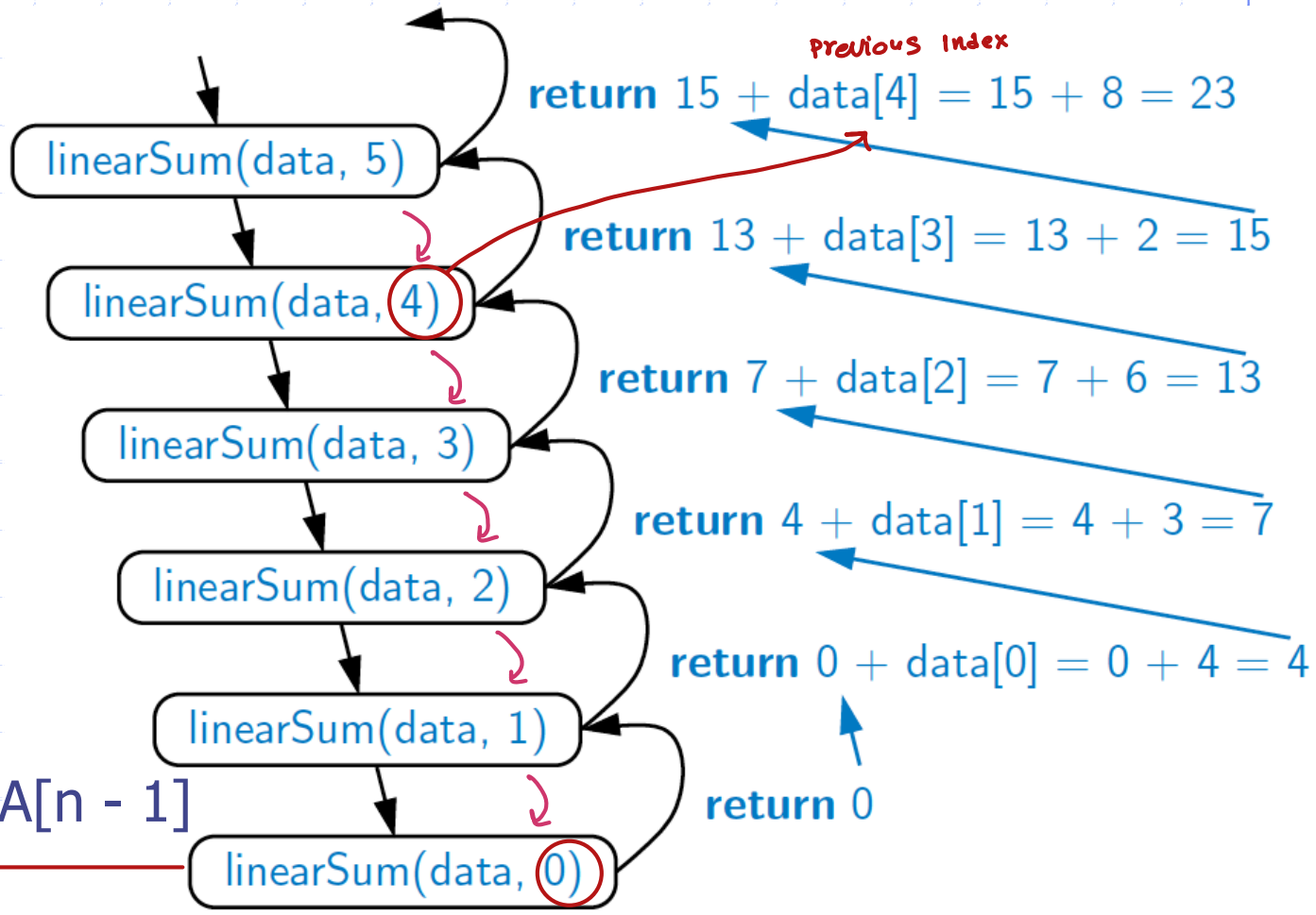
```
if ( i < first)
   return ;

else

{

   linear Print ( data , first , last , i-1 );

   System.out.Println ( data [i] );
```

Example of Linear Recursion

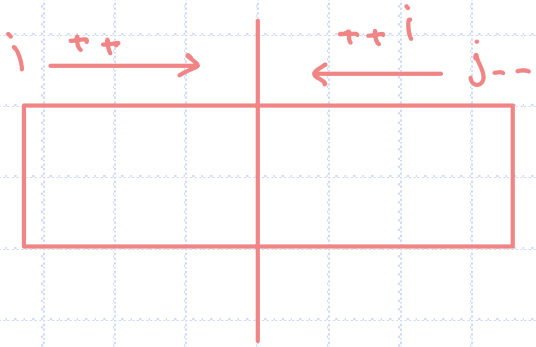
Algorithm linearSum(A, n):
Input:
  Array, A, of integers
  Integer n such that
    $0 \leq n \leq |A|$
Output:
    Sum of the first n integers in A

```
if n = 0 then
  return 0
else
  return
linearSum(A, n - 1) + A[n - 1]
```

Recursion trace of linearSum(data, 5) called on array data = [4, 3, 6, 2, 8]



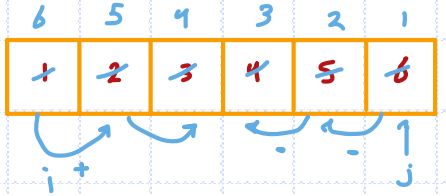
Reversing an Array



Algorithm reverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at



if i < j then

Swap A[i] and A[j]

reverseArray(A, i + 1, j – 1)

return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional parameters that are passed to the method.
- For example, we defined the array reversal method as reverseArray(A, i, j), not reverseArray(A)

must update it.

```
1  /** Reverses the contents of subarray data[low] through data[high] inclusive. */
2  public static void reverseArray(int[ ] data, int low, int high) {
3    if (low < high) {                                  // if at least two elements in subarray
4      int temp = data[low];                            // swap data[low] and data[high]
5      data[low] = data[high];
6      data[high] = temp;
7      reverseArray(data, low + 1, high − 1);           // recur on the rest
8    }
9  }
```

True

الي حطيتها صفا صي الي ابدا فيها في temp .

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \times p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls)
- We can do better than this, however

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$
$$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$
$$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$
$$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128$$

Recursive Squaring Method

Algorithm Power(x, n):

Input: A number x and integer n = 0

Output: The value x^n

if n = 0 **then**

return 1

if n is odd **then**

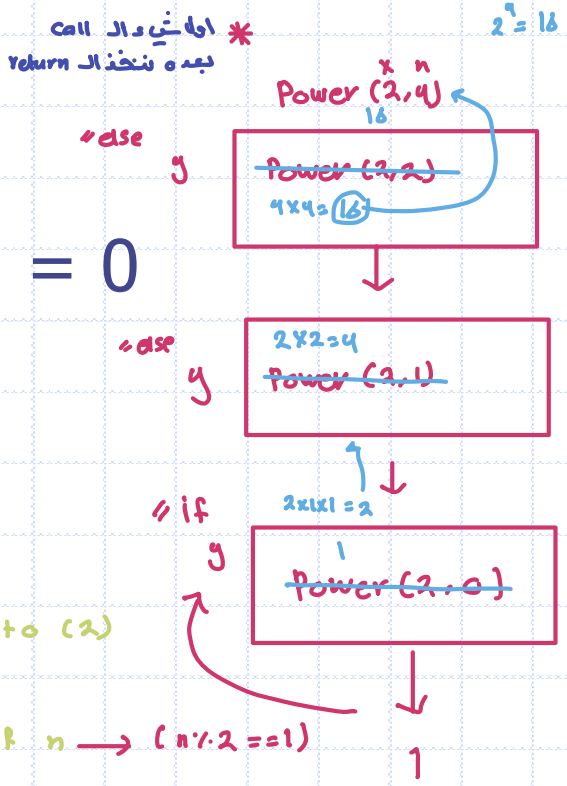
$y = \text{Power}(x, (n - 1)/2)$

return $x \cdot y \cdot y$

else

$y = \text{Power}(x, n/2)$

return $y \cdot y$



Analysis

```
Algorithm Power(x, n):
    Input: A number x and
  integer n = 0
    Output: The value x^n
   if n = 0    then
        return 1
   if n is odd then
        y  = Power(x, (n - 1)/ 2)
        return x · y · y
   else
        y = Power(x, n/ 2)
        return y · y
```

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

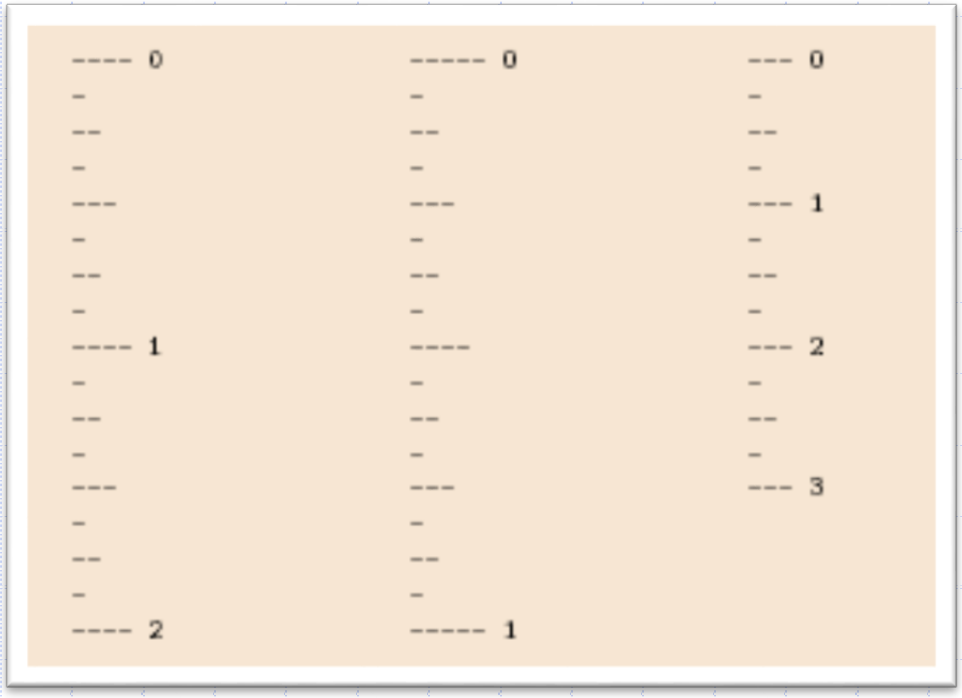
Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

```
Algorithm IterativeReverseArray(A, i, j ):
    Input: An array A and nonnegative integer indices i and j
    Output: The reversal of the elements in A starting at
  index i and ending at j
   while i <  j do
      Swap A[i ] and A[ j ]
      i  = i + 1
      j  = j - 1
   return
```

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example from before: the drawInterval method for drawing ticks on an English ruler.

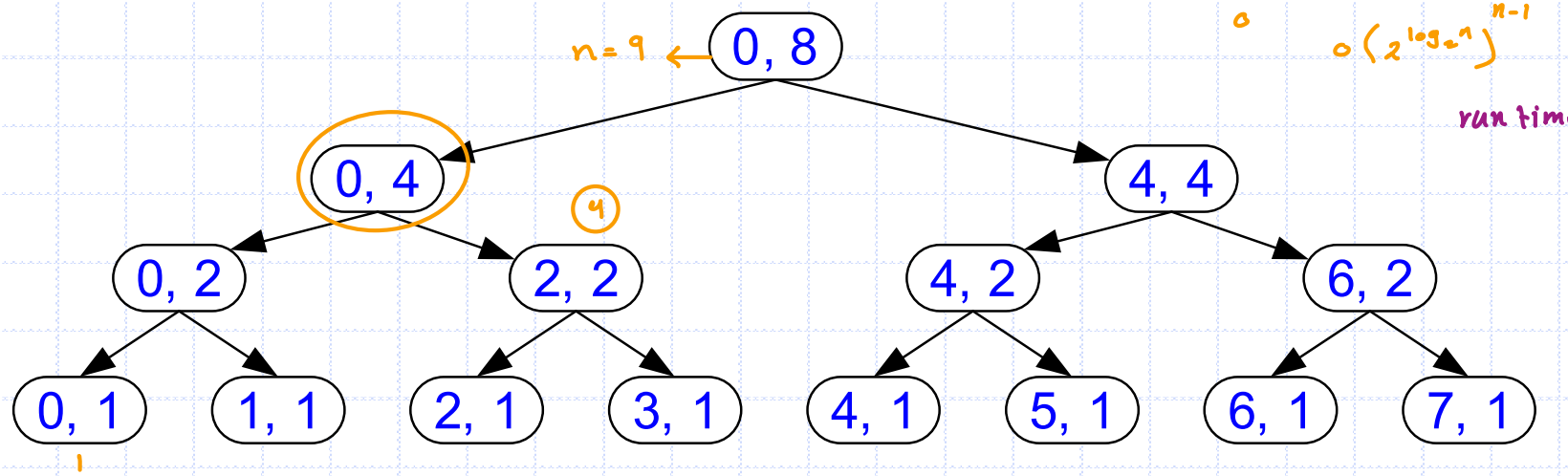


Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

```
Algorithm BinarySum(A, i, n):
    Input: An array A and integers i and n
    Output: The sum of the n integers in A starting at index i
    if n = 1 then
    return A[i]
    return BinarySum(A, i, n/ 2) + BinarySum(A, i + n/ 2, n/ 2)
```

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

- Recursive algorithm (first attempt):

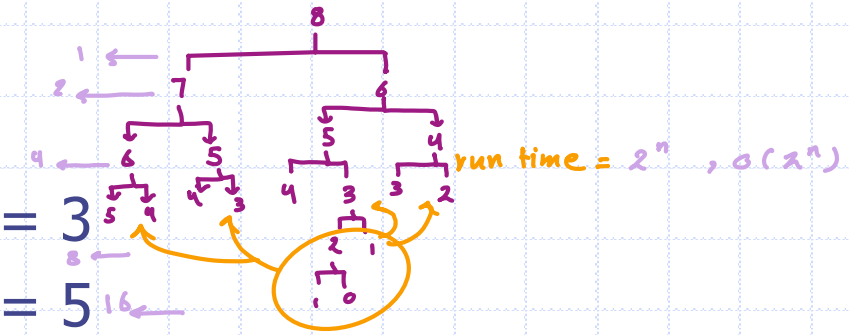
when we stop = 0

Algorithm BinaryFib(k):
  Input: Nonnegative integer k
  Output: The kth Fibonacci number F_k
  if $k = 1$ **then**
    return k
  else
    return BinaryFib($k - 1$) + BinaryFib($k - 2$)

if k=0 then return 0 else

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67.$



- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

Algorithm LinearFibonacci(k):
  Input: A nonnegative integer k
  Output: Pair of Fibonacci numbers (F_k, F_{k-1})
  if k = 1 **then**
    return (k, 0)
  else
    (i, j) = LinearFibonacci(k – 1)
    return (i +j, i)

- LinearFibonacci makes k–1 recursive calls

more than 2 call.

Multiple Recursion

- Motivating example:
  - summation puzzles
    - *pot* + *pan* = *bib*
    - *dog* + *cat* = *pig*
    - *boy* + *girl* = *baby*
- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

```
Algorithm PuzzleSolve(k,S,U):
  Input: Integer k, sequence S, and set U (universe of elements to
     test)
  Output:  Enumeration of all k-length extensions to S using elements
     in U without repetitions
  for all e  in U do
     Remove e from U      {e is now being used}
     Add e to the end of S
     if k = 1 then
           Test whether S is a configuration that solves the puzzle
           if S solves the puzzle then
                 return "Solution found: " S
     else
           PuzzleSolve(k - 1, S,U)
     Add e back to U       {e is now unused}
     Remove e from the end of S
```

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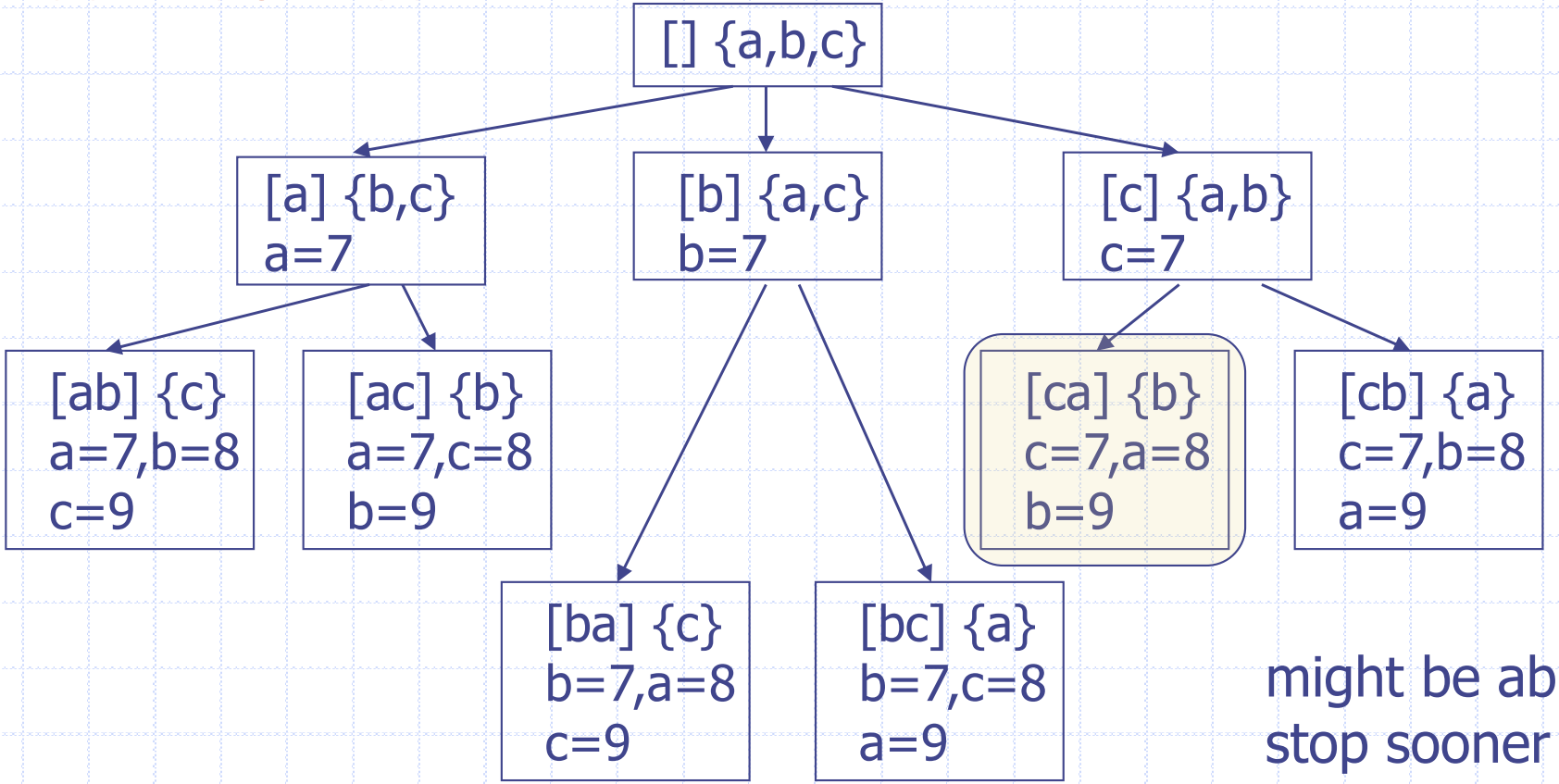
Example

cbb + ba = abc

799 + 98 = 997

a,b,c stand for 7,8,9; not necessarily in that order



might be able to stop sooner

Visualizing PuzzleSolve